Instructions:

1. Write your student number here:

2. Write your name here:

3. Write your student number ONLY (not your name) on the top of each page.

4. Start each problem on a new page. Use one side of the paper only. DO NOT work on the back of the page.

5. The exam is closed book, but you may have your 3 page double-sided notes.

6. You must hand in your notes, this exam packet, and your answers.

7. The number of pts assigned to each question is indicated before the question. The full score for the exam is 100.
1. (16 pts) Let \( X \) denote a random variable whose conditional distribution is Poisson given mean \( \lambda \), where \( \lambda \) is a random variable too, \( \lambda \sim \text{Gamma}(\alpha, \beta) \). The p.d.f. of gamma distribution is 
\[
f(\lambda) = \frac{\lambda^{\alpha-1}e^{-\lambda/\beta}}{\beta^\alpha \Gamma(\alpha)}, \text{ for } \lambda > 0 \text{ and } \alpha, \beta > 0.
\]

a. (8 pts) Show that the marginal p.m.f. of \( X \) is a negative binomial distribution of the form 
\[
P(X = x) = \binom{x + r - 1}{r - 1} p^r (1 - p)^x
\]
and express the parameters \( r \) and \( p \) in terms of \( \alpha, \beta \)

b. (8 pts) Given the moment generating function of \( X \) as 
\[
\left( \frac{p}{1 - (1 - p)e^t} \right)^r,
\]
calculate the mean and the variance of \( X \), expressed in terms of \( \alpha \) and \( \beta \).
2. (18 pts) Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables from a distribution with p.d.f. $f(x) = \frac{3x^2}{2\theta^3}$, of which the support is $-\theta \leq x \leq \theta$, with $\theta > 0$.

a. (6 pts) Let $Y = X^2$, find the p.d.f. and the support of $Y$.

b. (6 pts) Find the MLE of $\theta$ (you can use either the Xs or the Ys).

c. (6 pts) Find the p.d.f. of $Y(n) = \max(Y_1, \cdots, Y_n)$. 
3. (18 pts) Let $X_1, ..., X_n$ be iid from a $\text{Gamma}(\alpha, \beta)$ distribution with $\alpha$ known. The pdf of a gamma distribution is:

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty; \quad \alpha > 0, \beta > 0.$$ 

a. (4 pts) Show that the distribution of $T = \Sigma X_i$ is $\text{Gamma}(n\alpha, \beta)$ by computing its moment generating function.

b. (14 pts) Find the best unbiased estimator of $1/\beta$. (hint: look at the statistic $1/\Sigma X_i$.)
4. (15 pts) Let $X_1, \ldots, X_{10}$ be i.i.d random variables from Bernoulli($p$).

a. (10 pts) Find the most powerful test of size $\alpha = 0.0547$ of the hypotheses $H_0 : p = \frac{1}{2}$ versus $H_1 : p = \frac{1}{4}$. Find the power of this test. (You may use the provided table for binomial probabilities)

b. (5 pts) For testing $H_0 : p \leq \frac{1}{2}$ versus $H_1 : p > \frac{1}{2}$, find the size and the power function of the test that rejects $H_0$ if $\sum_{i=1}^{10} X_i \geq 6$. 
5. (18 pts) Suppose a Markov chain with state space \{0, 1\} has transition probability matrix

\[ P = \begin{bmatrix} (1 - a) & a \\ b & (1 - b) \end{bmatrix}, \]

with \(0 \leq a, b \leq 1\).

(a) (3 pts) Give conditions under which the chain is irreducible.

(b) (3 pts) Give conditions under which the chain is aperiodic.

(c) (6 pts) What is the limiting distribution of the chain (when \(0 < a, b < 1\))? Carefully describe the theoretical justification for the existence of this limiting distribution.

(d) (6 pts) Suppose the stationary distribution of this Markov chain is \(\pi\). Now assume you pick \(X_0 = 1\) to be the initial state of the Markov chain and run it forward using the above transition matrix \(P\) to obtain the Markov chain \(X_1, X_2, \ldots\). Let \(p_n\) be the proportion of times the Markov chain has been in state 0 by time \(n\). What does \(p_n\) converge to as \(n \to \infty\)? Clearly state the result (which kind of convergence) and the associated assumptions.
6. (15 pts) Consider a homogeneous Poisson process \( \{N(t), t > 0\} \) with intensity \( \lambda \).

(a) (8 pts) Prove, stating any properties of Poisson processes that you may have used, that given that \( N(t) = 1 \) (exactly one event between time 0 and time \( t \)), the distribution of the arrival time of that single event is uniformly distributed over \([0, t]\).

(b) (7 pts) Suppose items arrive at a processing plant in accordance with a homogeneous Poisson process with intensity \( \lambda \). Each day, the plant operates for a period of time \( T \). All items that arrive prior to time \( t \) \( (t < T) \) are dispatched at time \( t \). All items that arrive after time \( t \) are dispatched at time \( T \).

i. (4 pts) What is the expected total wait of all items arriving prior to time \( t \)?

ii. (3 pts) What is the expected total wait of all items that arrive in a day?