**Theorem:** $S(T) = \int_0^T \frac{\sin x}{x} \, dx$ is bounded in $T$ and $S(T) \to \frac{\pi}{2}$ as $T \to \infty$. (Note that $\frac{\sin x}{x} = 1$ at $x = 0$).

**Proof:** As $\frac{1}{x} = \int_0^\infty e^{-ux} \, du$ for $x > 0$, we get by using Fubini’s theorem to interchange the order of integration that

$$S(T) = \int_0^T \sin x \left( \int_0^\infty e^{-ux} \, du \right) \, dx = \int_0^\infty J_T(u) \, du, \quad (1)$$

where $J_T(u) = \int_0^T e^{-ux} \sin x \, dx$. Integration by parts yields,

$$J_T(u) = \sin x \frac{e^{-ux}}{-u} \bigg|_0^T + \frac{1}{u} \int_0^T e^{-ux} \cos x \, dx = -\sin T \frac{e^{-uT}}{u} + \frac{1}{u} \int_0^T e^{-ux} \cos x \, dx. \quad (2)$$

Another application of integration by parts yields,

$$\int_0^T e^{-ux} \cos x \, dx = \cos x \frac{e^{-ux}}{-u} \bigg|_0^T - \int_0^T (-\sin x) \frac{e^{-ux}}{-u} \, dx$$

$$= -\cos T \frac{e^{-uT}}{u} + \frac{1}{u} - \frac{1}{u} \int_0^T e^{-ux} \sin x \, dx$$

$$= \frac{1}{u} - \cos T \frac{e^{-uT}}{u} - \frac{1}{u} J_T(u). \quad (3)$$

By (2) and (3),

$$J_T(u) \left( 1 + \frac{1}{u^2} \right) = \frac{1}{u^2} - \frac{1}{u} e^{-uT} \sin T - \frac{1}{u^2} e^{-uT} \cos T.$$

So

$$J_T(u) = \frac{1}{1 + u^2} - \frac{e^{-uT}}{1 + u^2} (\cos T + u \sin T) = O \left( \frac{1}{1 + u^2} \right), \quad (4)$$

hence by (1), we have for some constant $c > 0$,

$$|S(T)| = \left| \int_0^T \frac{\sin x}{x} \, dx \right| \leq \int_0^\infty |J_T(u)| \, du \leq c \int_0^\infty \frac{1}{1 + u^2} \, du < \infty.$$

By (4) and Lebesgue’s dominated convergence theorem,

$$S(T) = \int_0^T \frac{\sin x}{x} \, dx = \int_0^\infty J_T \, du \to \int_0^\infty \frac{1}{1 + u^2} \, du = \frac{\pi}{2}.$$
Second proof using Cauchy’s Theorem

Use the closed path $\Gamma_{\delta,T} = \Gamma_{1,\delta,T} \cup \Gamma_{2,\delta} \cup \Gamma_{3,\delta,T} \cup \Gamma_{4,T}$

By Cauchy’s theorem for closed paths,

$$
\int_{\Gamma_{\delta,T}} \frac{e^{iz}}{z} = 0 \text{ for all } T, \delta > 0. \quad (a)
$$

Note that with $\theta \in (0, \pi)$,

$$
z = \begin{cases}
    Te^{i\theta} & \text{on } \Gamma_{4,T}, \\
    \delta e^{i\theta} & \text{on } \Gamma_{2,\delta},
\end{cases}
$$

and $\frac{1}{z} \frac{dz}{d\theta} = i$ on $\Gamma_{\delta,T}$. Since $|e^{i(x+iy)}| \leq e^{-y}$, and $e^{-T\sin\theta} \to 0$ when $\sin\theta > 0$ as $T \to \infty$,

$$
\left| \int_{\Gamma_{4,T}} \frac{e^{iz}}{z} dz \right| \leq \int_{0}^{\pi} |Te^{i\theta}| \, d\theta \leq \int_{0}^{\pi} e^{-T\sin\theta} \, d\theta \to 0 \quad (b)
$$

by Lebesgue’s dominated convergence theorem. Another application of the dominated convergence theorem gives,

$$
\int_{\Gamma_{2,\delta}} \frac{e^{iz}}{z} dz = \int_{\pi}^{0} \frac{e^{\delta e^{i\theta}}}{e^{i\theta}} \, i e^{i\theta} \, d\theta = -i \int_{0}^{\pi} e^{i\delta e^{i\theta}} \, d\theta \to -i\pi \text{ as } \delta \to 0. \quad (c)
$$

Further,

$$
\int_{\Gamma_{1,\delta,T} \cup \Gamma_{3,\delta,T}} \frac{e^{iz}}{z} dz = \left( \int_{-T}^{-\delta} + \int_{\delta}^{T} \right) \left( \cos \frac{x}{x} + i \frac{\sin x}{x} \right) \, dx = 2i \int_{\delta}^{T} \frac{\sin x}{x} \, dx, \quad (d)
$$

as $\frac{\sin x}{x}$ is an even function and $\frac{\cos x}{x}$ is an odd function. So by (a)-(d),

$$
\int_{\delta}^{T} \frac{\sin x}{x} \, dx \to \frac{\pi}{2},
$$

as $\delta \to 0, T \to \infty$. 

---

**Diagram:**

- $\Gamma_{1,\delta,T}$
- $\Gamma_{2,\delta}$
- $\Gamma_{3,\delta,T}$
- $\Gamma_{4,T}$
- $-T$,
- $-\delta$,
- $0$,
- $\delta$,
- $T$,
- $y$,
- $x$.