Integrals of general measurable functions

If $f$ is measurable, then both $f^+ = \max(0, f)$, $f^- = \max(0, -f)$ are measurable, and $f = f^+ - f^-$, $|f| = f^+ + f^-$. If $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$, then $\int f(\omega)d\mu(\omega) = \int f(\omega)\mu(d\omega) = \int f d\mu = \int f^+ d\mu - \int f^- d\mu$. This is called definite integral of $f$.

**Definition:** $f$ is integrable if $\int |f| d\mu < \infty$.

In this case, $\int f^+ d\mu + \int f^- d\mu = \int |f| d\mu < \infty$. So $f$ is integrable if and only if both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite.

**Result:** $f$ has definite integral and $f = g$ a.e. $\Rightarrow$ $\int f d\mu = \int g d\mu$.

**Proof:** $f^+ = g^+$ a.e., $f^- = g^-$ a.e. So $\int f^\pm d\mu = \int g^\pm d\mu$.

**Theorem:** Suppose $f$, $g$ are integrable

i) $f \leq g$ a.e. $\Rightarrow\int f d\mu \leq \int g d\mu$.

**Proof:** $f \leq g$ a.e. $\Rightarrow f^+ \leq g^+$ a.e., $g^- \leq f^-$ a.e. As $\int f^+ d\mu \leq \int g^+ d\mu < \infty$, $\int g^- d\mu \leq \int f^- d\mu < \infty$, we have $\int f d\mu \leq \int g d\mu$.

ii) **Linearity:** $a$ and $b$ finite real numbers, then $(af + bg)$ is integrable and $\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$.

**Proof:** $af + bg$ is integrable as $|af + bg| \leq |a||f| + |b||g|$. For $a > 0$, $\int a f d\mu = \int (af)^+ d\mu - \int (af)^- d\mu = a \int f^+ d\mu - a \int f^- d\mu = a \int f d\mu$.

Similarly for $a < 0$. So enough to prove the result for $a = b = 1$.

Note that $(f + g)^+ - (f + g)^- = f + g = f^+ + g^+ - f^- - g^-$ implies $(f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+$.

As these functions are all non-negative, $\int (f + g)^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int (f + g)^- d\mu + \int f^+ d\mu + \int g^+ d\mu$. A rearrangement of this completes the proof.

iii) $|\int f d\mu| \leq \int |f| d\mu$ and $\int f d\mu - \int g d\mu| \leq \int |f - g| d\mu$.

**Proof:** By i), ii), $-|f| \leq f \leq |f|$ $\Rightarrow -\int |f| d\mu \leq \int f d\mu \leq \int |f| d\mu$. 
Examples

(a) $\mathcal{F} = \text{all subsets of } \Omega = \{1, 2, \ldots \}$ and $\mu$ is the counting measure. A sequence $\{x_n\}$ of real numbers can be considered as a measurable function $f$, $f(n) = x_n$. So $f$ is integrable if and only if the sequence $\{x_n\}$ is absolutely convergent. The sequence given by $x_m = (-1)^m \frac{1}{m}$ is not integrable.

(b) $\Omega, \mathcal{F}, \mu$ as above. Let $f \equiv 0$, $f_n = I_{\{n, n+1, \ldots \}}$. Then $\lim_n f_n(m) \to f(m)$ for all $m$, but $\int f \, d\mu = 0$ and $\int f_n \, d\mu = \infty$ for all $n$.

(c) $f$ is a Borel function, bounded on bounded intervals. Both $f_n = fl((-n, n))$, $g_n = fl((-n, n+1))$ converge point-wise to $f$. Even if the limits $\lim_n \int f_n \, d\lambda$ and $\lim_n \int g_n \, d\lambda$ exist, they may not be equal; e.g. $f(x) = x$.

(d) $f = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k} I_{(k, k+1]}$ has no integral even though the $\lim_n \int fl((-n, n)) d\lambda$ exists.

(e) Let $f_n = n^2 I_{(0, n-1)}$, $f \equiv 0$. Then $f_n \to f$ everywhere, $\int fd\lambda = 0$ and $\int f_n \, d\lambda = n \to \infty$.

Convergence of Integrals

Does $f_n \to f$ a.e. imply $\int f_n \, d\mu \to \int f \, d\mu$? Not always (see (b), (e)).

Monotone Convergence Theorem: $0 \leq f_n \uparrow f$ a.e. $\Rightarrow$ $\int f_n \, d\mu \uparrow \int f \, d\mu$.

Proof: There exists a measurable set $A$ such that $\mu(A^c) = 0$ and $0 \leq f_n I_A \uparrow f I_A$. So $\int f_n \, d\mu = \int f_n I_A \, d\mu \to \int f I_A \, d\mu = \int f \, d\mu$.

Examples:

1. For each $m$, $0 \leq x_{nm} \uparrow x_m$ as $n \to \infty$ $\Rightarrow$ $\lim_n \sum_m x_{nm} = \sum_m x_m$. (Examples 16.1, 16.3).
   False for decreasing limits. Let $y_{nm} = I_{\{m \geq n\}}$ and $y_m = 0$.
   Then $0 \leq y_{nm} \downarrow y_m$ but $\sum_m y_{nm} = \infty \neq 0 = \sum_m y_m$. (See (b)). (Toeplitz lemma)

2. $(\Omega, \mathcal{F}, \mu)$, $\mathcal{F}_0$ is a sub $\sigma$-field of $\mathcal{F}$. The measure $\mu_0$ on $\mathcal{F}_0$ is the restriction of $\mu$. $f$ is measurable $\mathcal{F}_0$. Then $\int f \, d\mu = \int f \, d\mu_0$.
   Holds for $f = I_A$ for $A \in \mathcal{F}_0$. So holds for simple functions; increasing limits of non-negative $f$, and for all measurable functions $f$, where at least one of $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ is finite.

3. $\int f \, d\mu = \sum_n \int f \, d\mu_n$ for $f \geq 0$ and $\mu, \mu_n$ are measures, and $\mu(A) = \sum_n \mu_n(A)$, for $A \in \mathcal{F}$. (Example 16.5)
**Theorem** (Fatou’s Lemma): For $f_n \geq 0$,

$$\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu.$$

**Proof**: If $g_n = \inf_{k \geq n} f_k$, then $0 \leq g_n \uparrow g = \liminf_n f_n$.

So $\int f_n \, d\mu \geq \int g_n \, d\mu \rightarrow \int g \, d\mu$.

**Theorem** (Lebesgue’s dominated convergence theorem):
If $|f_n| \leq g$ a.e. for some integrable $g$ and if $f_n \rightarrow f$ a.e., then $f$ and $f_n$ are integrable and $\lim_n \int f_n = \int \lim f_n \, d\mu = \int f \, d\mu$.

**Proof**: Since $g + f_n$ and $g - f_n$ are non-negative, Fatou’s lemma gives

$$\int g \, d\mu + \int \liminf_n f_n \, d\mu = \int \liminf_n (g + f_n) \, d\mu \leq \liminf_n \int (g + f_n) \, d\mu = \int g \, d\mu + \liminf_n \int f_n \, d\mu$$

and

$$\int g \, d\mu - \int \limsup_n f_n \, d\mu = \int \liminf_n (g - f_n) \, d\mu \leq \liminf_n \int (g - f_n) \, d\mu = \int g \, d\mu - \limsup_n \int f_n \, d\mu.$$

So

$$\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu \leq \limsup_n \int f_n \, d\mu \leq \int \limsup_n f_n \, d\mu.$$

**Properties of Integrals**

1. *The Weierstrass M-test for series*: Suppose for each $m$, $x_{nm} \rightarrow x_m$ as $n \rightarrow \infty$ and $|x_{nm}| \leq M_m$. If $\sum M_m < \infty$, then $\sum x_{nm} \rightarrow \sum x_m$.

2. *Bounded convergence theorem*: If $\mu$ is a finite measure and the $f_n$ are uniformly bounded, then $f_n \rightarrow f$ a.e. implies $\int f_n \, d\mu \rightarrow \int f \, d\mu$.

3. If $f_n \geq 0$, then $\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$. ($\sum_{n=1}^k f_n \uparrow \sum_n f_n$).

4. If $\sum_n f_n$ converges a.e. and $|\sum_{k=1}^n f_k| \leq g$ for some integrable $g$, then $\sum_n f_n$ and $f_n$ are integrable and $\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$.

5. If $\sum_n \int |f_n| \, d\mu < \infty$, then $\sum_n f_n$ converges absolutely a.e. and is integrable, and $\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$.

6. Integral of $f$ over a set is defined as $\int_A f \, d\mu = \int I_A f \, d\mu$. If $A_1, A_2, \ldots$ are disjoint, and if $f$ is either non-negative or integrable, then $\int_{\bigcup_n A_n} f \, d\mu = \sum_n \int_{A_n} f \, d\mu$. 

If $A_1, A_2, \ldots$ are disjoint, and if $f$ is either non-negative or integrable, then $\int_{\bigcup_n A_n} f \, d\mu = \sum_n \int_{A_n} f \, d\mu$. 

**Definition:** \( \{ f_n \} \) is uniformly integrable (u.i.) if

\[
\lim_{a \to \infty} \sup_n \int_{|f_n| \geq a} |f_n| \, d\mu = 0.
\]

If \( \sup_n \int |f_n|^\beta \, d\mu < \infty \) for some \( \beta > 1 \), then \( \{ f_n \} \) is u.i.

If \( \{ f_n \}, \{ g_n \} \) are uniformly integrable, then so is \( \{ f_n + g_n \} \). Use

\[
\int_{|f+g| \geq 2a} |f + g| \, d\mu \leq 2 \int_{|h| \geq a} h \, d\mu \leq 2 \int_{|f| \geq a} |f| \, d\mu + 2 \int_{|g| \geq a} |g| \, d\mu,
\]

where \( h = \max(|f|, |g|) \).

**Theorem:** Suppose \( \mu \) is a finite measure and \( f_n \to f \) a.e.

a) If \( \{ f_n \} \) is u.i., then \( f \) is integrable and \( \int f_n \, d\mu \to \int f \, d\mu \).

b) If \( f, f_n \) are non-negative and integrable, then \( \int f_n \, d\mu \to \int f \, d\mu \) implies \( \{ f_n \} \) is u.i.

**Proof of a):** Integrability of \( f \) follows from Fatou's lemma and

\[
\int |f_n| \, d\mu \leq a\mu(\Omega) + \int_{|f_n| \geq a} |f_n| \, d\mu.
\]

If \( \mu(|f| = a) = 0 \), then \( f_n \mathbb{1}_{|f_n| < a} \to f \mathbb{1}_{|f| < a} \) a.e. and by bounded convergence theorem

\[
\int f_n \mathbb{1}_{|f_n| < a} \, d\mu \to \int f \mathbb{1}_{|f| < a} \, d\mu.
\]

Use u.i.

**Proof of b):** If \( \mu(|f| = a) = 0 \), then \( \int f_n \mathbb{1}_{|f_n| \geq a} \, d\mu \to \int f \mathbb{1}_{|f| \geq a} \, d\mu \). The result follows as \( f \) is integrable.

**Uniformly Integrability (continued)**

**Theorem:** Suppose \( \mu \) is a finite measure, \( f, f_n \) are integrable and \( f_n \to f \) a.e. The following are equivalent:

i) \( \{ f_n \} \) is u.i.

ii) \( \int |f_n - f| \, d\mu \to 0 \)

iii) \( \int |f_n| \, d\mu \to \int |f| \, d\mu \).

**Proof:** i) implies u.i. of \( \{|f_n - f|\} \). So ii) follows by the above theorem.

ii) implies iii) as \( ||f_n| - |f|| \leq |f_n - f| \).

i) follows from iii) using the theorem above.

**Note:** If \( |f_n| \leq g \) for some integrable \( g \), then \( \{ f_n \} \) is u.i.

\( f_n = (n/ \log n) l_{(0,n^{-1})}, \ n \geq 3 \) are u.i but not dominated by any integrable \( g \).
Let \( \{ f_t, t > 0 \} \) be a family of measurable functions and 
\[ \lim_{t \to \infty} f_t(\omega) = f(\omega) \] for \( \omega \in A \) and some measurable function \( f \), 
where \( A \) is a measurable set and \( \mu(A^c) = 0 \). If \( |f_t(\omega)| \leq g(\omega) \) for 
\( \omega \in A \) and \( g \) is integrable, then \( \int f_t \, d\mu \to \int f \, d\mu \) as \( t \to \infty \).

**Theorem:** Suppose that \( f(\omega, t) \) is measurable and integrable function of \( \omega \) for each \( t \in (a, b) \). Let 
\[ \ell(t) = \int f(\omega, t) \mu(d\omega). \]

(i) Suppose \( A \) is measurable, \( \mu(A^c) = 0 \) and for each \( \omega \in A \), \( f(\omega, t) \) is continuous in \( t \) at \( t_0 \); suppose further 
\[ |f(\omega, t)| \leq g(\omega) \] for \( \omega \in A \) and \( |t - t_0| < \delta \), where \( \delta \) does not depend on \( \omega \) and \( g \) is integrable. Then \( \ell \) is continuous at \( t_0 \).

(ii) Suppose \( A \) is measurable, \( \mu(A^c) = 0 \) and for each \( \omega \in A \), \( f(\omega, t) \) is differentiable in \( (a, b) \) with a derivative \( f'(\omega, t) \); 
suppose further \( |f'(\omega, t)| \leq g(\omega) \) for \( \omega \in A \) and \( t \in (a, b) \), where \( g \) is integrable. Then \( \ell \) has derivative \( \int f'(\omega, t) \mu(d\omega) \) on \( (a, b) \).

**Proof of (ii):** By mean-value theorem, for \( \omega \in A \) and for some \( s \) between \( t \) and \( t + h \), 
\[ \frac{1}{h} [f(\omega, t + h) - f(\omega, t)] = f'(\omega, s) \quad \text{which} \to f'(\omega, t). \]

By dominated convergence theorem, 
\[ \frac{1}{h} [\ell(t+h) - \ell(t)] = \int \frac{1}{h} [f(\omega, t+h) - f(\omega, t)] \mu(d\omega) \to \int f'(\omega, t) \mu(d\omega). \]

**Note:** Condition on \( g \) can be slightly weakened. Enough to assume that for each \( t \), there is an integrable \( g(\omega, t) \) such that 
\[ |f'(\omega, s)| \leq g(\omega, t) \] for \( \omega \in A \) and all \( s \) in some neighborhood of \( t \).
Integration over sets

Recall that integral of $f$ over a set is defined as $\int_A f \, d\mu = \int I_A f \, d\mu$. If $f$ is non-negative, then $\nu(A) = \int_A f \, d\mu$ defines a measure on $\mathcal{F}$.

**Theorem:**

I. Suppose $f, g$ are non-negative and $\int_A f \, d\mu = \int_A g \, d\mu$ for all $A \in \mathcal{F}$. If $\mu$ is $\sigma$-finite, then $f = g$ a.e.

II. $f, g$ are integrable and $\int_A f \, d\mu = \int_A g \, d\mu$ for all $A \in \mathcal{F}$ $\Rightarrow$ $f = g$ a.e.

III. If $f, g$ are integrable and $\int_A f \, d\mu = \int_A g \, d\mu$ for all $A \in \mathcal{P}$, where $\mathcal{P}$ is a $\pi$-system generating $\mathcal{F}$ and $\Omega$ is a finite or countable union of $\mathcal{P}$-sets, then $f = g$ a.e.

**Proof:** Suppose $\int_A f \, d\mu \leq \int_A g \, d\mu$ for all $A \in \mathcal{F}$ (a).

If $A_n \in \mathcal{F}$, $A_n \uparrow \Omega$, $\mu(A_n) < \infty$, $B_n = \{0 \leq g < f, \; g < n\}$, then $\int_{A_n \cap B_n} (f - g) \, d\mu \geq 0$, and by (a), $\int_{A_n \cap B_n} f \, d\mu \leq \int_{A_n \cap B_n} g \, d\mu < \infty$. So $\int_{A_n \cap B_n} (f - g) \, d\mu = 0$, which implies $(f - g)I_{A_n \cap B_n} = 0$ a.e. Thus $\mu(A_n \cap B_n) = 0$ and hence $\mu(0 \leq g < f, \; g < \infty) = 0$. So $f \leq g$ a.e. This proves I.

If $f, g$ are integrable, and (a) holds, then $\int I_{\{g < f\}} (f - g) \, d\mu = 0$ and hence $\mu(g < f) = 0$. This leads to II.

III follows from II and an earlier result.

**Densities**

A non-negative function $\delta$ is called the *density* of $\nu$ with respect to $\mu$ if $\nu(A) = \int_A \delta \, d\mu$, for all $A \in \mathcal{F}$.

**Theorem:** If $\nu$ has density $\delta$ with respect to $\mu$, then $\int f \, d\nu = \int f \, \delta \, d\mu$ holds for all non-negative measurable $f$.

A measurable function $g$ is integrable with respect to $\nu$ if and only if $g\delta$ is integrable with respect to $\mu$, in which case for all $A \in \mathcal{F}$, \[
\int_A g \, d\nu = \int_A g \delta \, d\mu. \tag{*}
\]

Prove (\textasteriskcentered) for $g = I_B$, $B \in \mathcal{F}$, then simple functions, non-negative functions, and finally for integrable functions.
Scheffe’s Theorem

\( \nu, \nu' \) are two finite measures with densities \( \delta, \delta' \) satisfying \( \nu(\Omega) = \nu'(\Omega) \).

\[
\nu'(A) - \nu(A) = \int_A (\delta' - \delta) \, d\mu = - \int_{A^c} (\delta' - \delta) \, d\mu
\]

\[
2|\nu'(A) - \nu(A)| = \left| \int_A (\delta' - \delta) \, d\mu \right| + \left| \int_{A^c} (\delta' - \delta) \, d\mu \right| \leq \int |\delta' - \delta| \, d\mu
\]

\[
2|\nu'(B) - \nu(B)| = \int_B |\delta' - \delta| \, d\mu + \int_{B^c} |\delta' - \delta| \, d\mu = \int |\delta' - \delta| \, d\mu,
\]

for all \( A \in \mathcal{F} \), where \( B = [\omega : \delta(\omega) < \delta'(\omega)] \).

**Theorem**: Suppose that \( \delta, \delta_n \) are densities of \( \nu, \nu_n \) with respect to \( \mu \). If \( \nu(\Omega) = \nu_n(\Omega) < \infty \) for all \( n \), and \( \delta_n \to \delta \) except on a set of \( \mu \)-measure zero, then \( \sup_{A \in \mathcal{F}} |\nu_n(A) - \nu(A)| = \frac{1}{2} \int |\delta_n - \delta| \, d\mu \to 0 \).

**Proof**: Let \( g_n = \delta - \delta_n \). As \( 0 \leq g_n^+ \leq \delta \), the dominated convergence theorem implies

\[
\int |g_n| \, d\mu \leq \int_{g_n \geq 0} g_n \, d\mu - \int_{g_n < 0} g_n \, d\mu = 2 \int_{g_n \geq 0} g_n \, d\mu = 2 \int g_n^+ \, d\mu \to 0.
\]

**Change of Variable**

\( T : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}') \) measurable transform.

\( \mu T^{-1}(A') = \mu(T^{-1}A') \), \( A' \in \mathcal{F}' \).

If \( f \) is non-negative, then

\[
\int_{\Omega} f(T \omega) \mu(d\omega) = \int_{\Omega'} f(\omega') \mu T^{-1}(d\omega').
\]

A measurable function \( g \) is integrable with respect to \( \mu T^{-1} \) if and only if \( gT \) is integrable with respect to \( \mu \), in which case for all \( A' \in \mathcal{F}' \),

\[
\int_{T^{-1}A'} g(T \omega) \mu(d\omega) = \int_{A'} g(\omega') \mu T^{-1}(d\omega').
\]