Lebesgue measure is translation invariant on Borel $\sigma$-field

$\Omega = (0, 1]$. $\lambda$ Lebesgue measure on the Borel $\sigma$-field $\mathcal{B}$.

$x \oplus y = x + y$ or $x + y - 1$ according as $x + y \in (0, 1]$ or not.

$A \oplus x = \{a \oplus x : a \in A\}$

$\mathcal{L} = \{A \in \mathcal{B} : A \oplus x \in \mathcal{B}$ and $\lambda(A \oplus x) = \lambda(A)$ for all $x \in \Omega\}$

Since $(A \oplus x)^c = A^c \oplus x$, $\mathcal{L}$ is a $\lambda$-system containing $\mathcal{I}$.

By $\pi - \lambda$ theorem, $\mathcal{L} = \mathcal{B}$. Thus $\lambda$ is translation invariant on $\mathcal{B}$.

$\lambda^*$ is translation invariant

In fact $\lambda^*$ is translation invariant on all subsets of $\Omega$.

Let $\mathcal{B}_0$ be the Borel field generated by $\mathcal{I}$.

Thus for $A \in \mathcal{B}_0$, $A \oplus x \in \mathcal{B}_0$ and $\lambda(A \oplus x) = \lambda(A)$.

If $B \subset \bigcup_{i=1}^\infty A_i$, $A_i \in \mathcal{B}_0$, then $B \oplus x \subset \bigcup_{i=1}^\infty (A_i \oplus x)$. Hence

$$\lambda^*(B \oplus x) \leq \sum_{i=1}^\infty \lambda(A_i \oplus x) = \sum_{i=1}^\infty \lambda(A_i)$$

This implies

$$\lambda^*(B \oplus x) \leq \lambda^*(B).$$

As $B = (B \oplus x) \oplus (1 - x)$,

$$\lambda^*(B) = \lambda^*((B \oplus x) \oplus (1 - x)) \leq \lambda^*(B \oplus x).$$

Thus $\lambda^*$ is translation invariant on all subsets of $\Omega$. 
The sets in $\mathcal{M} = \mathcal{M}(\lambda^*)$ are called Lebesgue measurable sets. $\lambda^*$ (called Lebesgue measure) is a probability measure on $\mathcal{M}$.

If $A \in \mathcal{M}$, then $\lambda^*(AE) + \lambda^*(A^cE) \leq \lambda^*(E)$ for all $E \subset \Omega$.

For any $B \subset \Omega$,
\[(B \oplus x) \cap E = (B \oplus x) \cap ((E \oplus (1-x)) \oplus x) = (B \cap (E \oplus (1-x))) \oplus x).\]

It follows that
\[\lambda^*((B \oplus x) \cap E) = \lambda^*(B \cap (E \oplus (1-x))) \oplus x) = \lambda^*(B \cap (E \oplus (1-x))).\]

So $\lambda^*((A \oplus x) \cap E) + \lambda^*((A \oplus x)^c \cap E) \leq \lambda^*(E \oplus (1-x)) = \lambda^*(E)$.

Hence $A \oplus x \in \mathcal{M}$.

This establishes that the Lebesgue measure on $\mathcal{M}$ is translation invariant.

**Nonmeasurable set**

Define $x \sim y$ if $x \oplus r = y$ for some rational $r \in (0, 1]$.

The relation $\sim$ partition $\Omega$ into equivalence classes $\{A_\theta : \theta \in \Theta\}$.

$H$ consists of exactly one point from each class $A_\theta$.

Let $Q$ be the set of rationals in $\Omega$ and $H_r = H \oplus r$.

Then $\bigcup_{r \in Q} H_r = \Omega$, and $H_r \cap H_s = \emptyset$ for $r \neq s$ in $Q$.

Further, if $P$ is a translation invariant probability measure on all subsets of $\Omega$, then

\[1 = P(\bigcup_{r \in Q} H_r) = \sum_{r \in Q} P(H_r) = \infty \cdot P(H_{1/2}).\]

Impossible!

*There is no translation-invariant probability measure on all subsets of $(0, 1]$. Consequently, $\mathcal{M}(\lambda^*) \neq 2^{(0,1]}$. There exists a non-Lebesgue measurable set.*
Let $L \in \mathcal{M}$, and $\mathcal{B}_0$ be the Borel field.

Get $B_{i,n} \in \mathcal{B}_0$ such that for each $n$,

$$B_n = \bigcup_{i=1}^{\infty} B_{i,n} \supset L \text{ and } \lambda^*(L) > \lambda(B_n) - \frac{1}{n}.$$  

Note that $B_n \in \mathcal{B}$, the Borel $\sigma$-field, and $\lambda$ is the extended Lebesgue measure on $\mathcal{B}$.

If $B = \bigcap_{n=1}^{\infty} B_n$, then $L \subset B \in \mathcal{B}$ and $\lambda^*(L) > \lambda(B) - \frac{1}{n}$ for all $n$. Thus $\lambda^*(L) = \lambda(B)$.

Similarly get $D \in \mathcal{B}$ such that $L^c \subset D^c$ and $\lambda^*(L^c) = \lambda(D^c)$.

As $L \in \mathcal{M}$, it follows that $\lambda^*(L) = \lambda(D)$.

It follows that $D \subset L \subset B$, $D, B \in \mathcal{B}$, and $\lambda(D) = \lambda^*(L) = \lambda(B)$. Note that $\lambda(B - D) = 0$.

## Monotone Class Theorem

**Definition:** A class $\mathcal{M}$ of subsets of $\Omega$ is called monotone class if it is closed under countable monotone limits.

That is, $A_i \uparrow A$, and $A_i \in \mathcal{M}$ implies $A \in \mathcal{M}$ and $B_i \downarrow B$, and $B_i \in \mathcal{M}$ implies $B \in \mathcal{M}$.

**Theorem:** If a monotone class $\mathcal{M}$ contains a field $\mathcal{F}$, then $\mathcal{M} \supset \sigma(\mathcal{F})$.

**Proof:** Clearly, a monotone class which is a field is also a $\sigma$-field.

Let $\mathcal{M}_0$ be the smallest monotone class containing $\mathcal{F}$.

Note that $\Omega \in \mathcal{F} \subset \mathcal{M}_0$. Enough to prove that $\mathcal{M}_0$ is a field.

Let $\mathcal{M}_A = \{B \in \mathcal{M}_0 : A \cap B, A \cap B^c \text{ and } A^c \cap B \in \mathcal{M}_0\}$.

Clearly $\mathcal{M}_A$ is a monotone class, and if $A \in \mathcal{F}$, then $\mathcal{F} \subset \mathcal{M}_A$.

Hence $\mathcal{M}_0 \subset \mathcal{M}_A$ by minimality of $\mathcal{M}_0$; consequently $\mathcal{M}_0 = \mathcal{M}_A$.

So for $B \in \mathcal{M}_0$ and $A \in \mathcal{F}$, $A \cap B, A \cap B^c$ and $A^c \cap B \in \mathcal{M}_0$.

Hence $\mathcal{F} \subset \mathcal{M}_B$ and again by minimality $\mathcal{M}_0 = \mathcal{M}_B$.

If $A, B \in \mathcal{M}_0 = \mathcal{M}_A$, then $A \cap B, A \cap B^c$ and $A^c \cap B \in \mathcal{M}_0$. Take $A = \Omega$ to conclude $\mathcal{M}_0$ is closed under complementation. Thus $\mathcal{M}_0$ is a field.

**Example:** $\Omega \neq \emptyset$, $\mathcal{P} = \{\emptyset\} = \mathcal{M}$.

$\mathcal{P}$ is a $\pi$-system, $\mathcal{M}$ is a monotone class, but $\sigma(\mathcal{P}) = \{\emptyset, \Omega\} \not\subset \mathcal{M}$.  