Field/Algebra

Motivation for a Field
- Sample Space
- Events
- Classes of sets
- Recall the notation ∈, ∪, ∩

Definition (Field)
A class $\mathcal{F}$ of subsets of a non-empty set $\Omega$ is called a field if it satisfies:

a) $\emptyset \in \mathcal{F}$, $\Omega \in \mathcal{F}$

b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (closed under complementation.)

c) If $A \in \mathcal{F}$, $B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$ (closed under finite unions.)

The term algebra is often used to denote a field.
A field is a $\sigma$-field or $\sigma$-algebra if it is closed under countable unions.
That is, if $A_n \in \mathcal{F}$ a $\sigma$-field, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Examples of fields

1. $\{\emptyset, \Omega\}$
2. $\{\emptyset, \Omega, A, A^c\}$, $A \subset \Omega$
3. Finite - cofinite field.
4. Countable - cocountable $\sigma$-field.
5. $\mathcal{I} = \{(a, b] : 0 \leq a \leq b \leq 1\}$,
   $\mathcal{B}_0 = \{\emptyset$, finite disjoint unions of sets from $\mathcal{I}\}$ is a field on $(0, 1]$.

Let $\mathcal{A}$ be a class of subsets of a non-empty set $\Omega$.

a) $\bigcap_{\text{field } \mathcal{G} \supseteq \mathcal{A}} \mathcal{G} = f(\mathcal{A})$, the smallest field containing $\mathcal{A}$, is called the field generated by $\mathcal{A}$.

b) $\bigcap_{\sigma\text{-field } \mathcal{G} \supseteq \mathcal{A}} \mathcal{G} = \sigma(\mathcal{A})$, the smallest $\sigma$-field containing $\mathcal{A}$, is called the $\sigma$-field generated by $\mathcal{A}$.

$\mathcal{B} = \sigma(\mathcal{I}) = \sigma(\mathcal{B}_0) = \sigma(\mathcal{I}_0) = \sigma(\text{open sets}) = \sigma(\text{open intervals})$ is the Borel $\sigma$-field on $(0, 1]$, where $\mathcal{I}_0 = \{(a, b] \in \mathcal{I} : a, b$ rationals}.
Sets in $\mathcal{B}$ are called Borel sets.
Remark (Exercise 2.5)

\[ G = \left\{ \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij} : A_{ij} \text{ or } A_{ij}^c \in \mathcal{A}, \ \bigcap_{j=1}^{n_i} A_{ij} \text{ are disjoint} \right\} \]

is the smallest field \( f(\mathcal{A}) \) containing \( \mathcal{A} \).

**Proof:** Clearly \( \mathcal{A} \subset G \), and any field containing \( \mathcal{A} \) should contain \( G \). Enough to show that \( G \) is a field.

- \( G \) is closed under finite intersections.

\[
\bigcup_{j=1}^{n_i} A_{ij}^c = \bigcup_{j=1}^{n_i} \left( A_{ij}^c \cap \bigcap_{k=1}^{j-1} A_{ik} \right) \in G
\]

(the sets after the last union sign are disjoint).

- It follows that \( \left( \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij} \right)^c = \bigcap_{i=1}^{m} \bigcup_{j=1}^{n_i} A_{ij}^c \in G \).

- Hence \( f(\mathcal{A}) = G \).

Elementary properties of fields

- If \( \mathcal{A} \) consists of singleton sets, then \( f(\mathcal{A}) \) is the finite - cofinite field.
- \( f(\mathcal{A}) \subset \sigma(\mathcal{A}) \).
- \( f(\mathcal{A}) = \sigma(\mathcal{A}) \) if \( \mathcal{A} \) is finite.
- If \( \mathcal{A} \) is countable, then \( f(\mathcal{A}) \) is countable.
- If \( \mathcal{F}_1, \mathcal{F}_2 \) are fields then \( f(\mathcal{F}_1 \cup \mathcal{F}_2) = G \), where

\[
G = \left\{ \bigcup_{i=1}^{m} (A_i \cap B_i) : A_i \in \mathcal{F}_1, B_i \in \mathcal{F}_2, A_i \cap B_i \text{ are disjoint} \right\}.
\]

[G is closed under intersections & \( A^c \cap B^c = A^c \cup (A \cap B^c) \)]

- If \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \) fields, then \( \bigcup_{i=1}^{n} \mathcal{F}_n \) is a field. It need not be a \( \sigma \)-field even if \( \mathcal{F}_n \) are \( \sigma \)-fields.

**Example:** \( \Omega = \{1, 2, 3, \ldots \} \), \( \mathcal{F}_n = \sigma(\{ \{k\} : 1 \leq k \leq n \}) \), \( \bigcup_{n}^{n} \mathcal{F}_n \) is finite - cofinite field. It is not a \( \sigma \)-field.
• $\sigma(A)$ is countably generated (separable) if $A$ is countable.

• Borel $\sigma$-field $\mathcal{B} = \sigma(\{(a, b) : 0 \leq a \leq b \leq 1 \text{ rationals}\})$ is countably generated.

• $\mathcal{F} = \{A \subset (0, 1] : A \text{ or } A^c \text{ is countable}\}$ is not countably generated

  \textit{Proof:} Suppose $\mathcal{F} = \sigma(\{A_1, A_2, \ldots\}) = \sigma(\{B_1, B_2, \ldots\})$, where $B_i = A_i$ or $A_i^c$ if $A_i$ is countable or not. $\mathcal{F} = \sigma(\mathcal{A}_0)$, where $\Omega_0 = \bigcup_{i=1}^{\infty} B_i$ is countable, and $\mathcal{A}_0 = \{\{x\} : x \in \Omega_0\}$.

  Since $\mathcal{A}_0 \subset \mathcal{G} = \{B, B \cup \Omega_0^c : B \subset \Omega_0\} \subset \mathcal{F}$ and $\mathcal{G}$ is a $\sigma$-field, it follows that $\mathcal{G} = \mathcal{F}$. But if $y \in \Omega_0^c$, then $\{y\} \notin \mathcal{G}$. Contradiction. So $\mathcal{F}$ is not separable.

• If $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_2$ is countably generated, then $\mathcal{F}_1$ need not be countably generated.

  \textit{Example:} $\mathcal{F}_1 =$ countable - co-countable $\sigma$-field on $(0, 1]$, and $\mathcal{F}_2 = \mathcal{B}$, the Borel $\sigma$-field.