This closed-book final is worth 20 points. You have 110 minutes. You are allowed to use three sheets (double-sided) of your own notes. Write your answers on separate pages, and be sure that your name appears on each page you turn in.

On this exam, to “find the asymptotic distribution” of the sequence \( \{Z_n\} \) means to give sequences \( \{a_n\} \) and \( \{b_n\} \) and a non-constant random variable \( X \) such that \( b_n(Z_n - a_n) \xrightarrow{d} X \).

**Problem 1.** Suppose that for \( \theta > 2 \), \( Y_1, Y_2, \ldots \) are independent and identically distributed from a Pareto distribution with density function \( f_\theta(y) = \frac{\theta}{y^{\theta + 1}}, y > 1 \). This implies that

\[
E Y_i = \frac{\theta}{\theta - 1} \quad \text{and} \quad \text{Var} Y_i = \frac{\theta}{(\theta - 2)(\theta - 1)^2},
\]

and the distribution function is \( F(y) = 1 - y^{-\theta} \) for \( y > 1 \).

(a) \[3 \text{ points}\] Let \( M_n = \max_{1 \leq i \leq n} Y_i \). Find the asymptotic distribution of \( M_n \).

**Solution:** Taking \( b_n \) to be nonnegative, we get

\[
P[b_n(M_n - a_n) \leq t] = P[M_n \leq (t/b_n) + a_n] = \left[1 - \left(\frac{t}{b_n} + a_n\right)^{-\theta}\right]^n
\]
as long as \((t/b_n) + a_n > 1\). Thus, we may take \( a_n = 0 \) and \( b_n = n^{-1/\theta} \). With this choice, we find that \( t/b_n \rightarrow \infty \) as \( n \rightarrow \infty \), so \((t/b_n) + a_n > 1\) will certainly be satisfied for large enough \( n \) as long as \( t > 0 \). On the other hand, for \( t \leq 0 \), \((t/b_n) + a_n \leq 0\) for all \( n \). We conclude that

\[
P[n^{-1/\theta} M_n \leq t] = \left[1 - \frac{t^{-\theta}}{n}\right]^n \rightarrow \exp\{-t^{-\theta}\} \quad \text{for} \ t > 0.
\]

We may express this as \( n^{-1/\theta} M_n \xrightarrow{d} X \), where \( X \) has the c.d.f. \( F(x) = \exp\{-x^{-\theta}\} \) for \( x > 0 \). Alternatively, we may write \( n^{-1/\theta} M_n \xrightarrow{d} W^{-1/\theta} \) where \( W \) is standard exponential.

(b) \[3 \text{ points}\] Find maximum likelihood estimator of \( \theta \) and its asymptotic distribution.

**Solution:** Setting the derivative of the loglikelihood equal to zero gives

\[
\ell'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} \log Y_i = 0,
\]
which has the unique root \( \hat{\theta} = n / \sum_{i=1}^{n} \log Y_i \). As this is the only root of the likelihood equation and all of the regularity conditions guaranteeing the existence of an efficient root are satisfied, we know that

\[
\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N \left( 0, \frac{1}{I(\theta)} \right),
\]

where in this case we may find \( I(\theta) = 1/\theta^2 \) by differentiating the log-density twice. We conclude that

\[
\sqrt{n} \left( \frac{n}{\sum_{i=1}^{n} \log Y_i} - \theta \right) \xrightarrow{d} N(0, \theta^2).
\]

**Problem 2. [3 points]** Prove one of the following two statements regarding sequences of univariate random variables (you may choose):

(i) If \( X_n \xrightarrow{d} c \) for some constant \( c \), then \( X_n \xrightarrow{P} c \).

**Solution:** Let \( X = c \) be a constant random variable and take \( \epsilon > 0 \). We obtain

\[
P(|X_n - c| \leq \epsilon) \geq P(X_n \leq c + \epsilon) - P(X_n \leq c - \epsilon).
\]

Since both \( c + \epsilon \) and \( c - \epsilon \) are continuity points of the c.d.f. of \( X \), the right hand side converges to \( P(X \leq c + \epsilon) - P(X \leq c - \epsilon) \), which equals one.

(ii) If \( X_n \xrightarrow{a.s.} X \), then \( X_n \xrightarrow{P} X \).

**Solution:** Let \( \epsilon > 0 \) and define sequences of events \( A_n \) and \( B_n \) as

\[
A_n = \{|X_k - X| < \epsilon \text{ for all } k \geq n\} \quad \text{and} \quad B_n = \{|X_n - X| < \epsilon\}.
\]

We see that \( A_n \subset B_n \) and so \( P(A_n) \leq P(B_n) \). But \( X_n \xrightarrow{a.s.} X \) means \( P(A_n) \to 1 \) by definition, so in this case \( P(B_n) \to 1 \) also.

**Problem 3.** Suppose that \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) are two independent simple random samples, with

\[
P(X_i \leq t) = P(Y_j \leq t + \theta) = F(t)
\]

for some continuous differentiable distribution function \( F(t) \) with \( f(t) = F'(t) \), assumed to have at least four finite moments. In other words, the cdf of the \( Y_j \) is \( F(t - \theta) \), shifted to the right by \( \theta \) from the distribution of the \( X_i \). We wish to test \( H_0 : \theta = \theta_0 \) against \( H_1 : \theta > \theta_0 \).

Here, we assume that \( n \equiv n_k \) and \( m \equiv m_k \) for \( k = 1, 2, \ldots \) and that both \( n \) and \( m \) go to infinity as \( k \to \infty \) in such a way that \( m/N \to \rho \), where \( N \equiv N_k = m_k + n_k \) and \( \rho \) is a positive constant less than 1.

Let

\[
S_k^2 = \frac{m}{N} \left[ \frac{1}{m} \sum_{i=1}^{m} (X_i - \bar{X}_m)^2 \right] + \frac{n}{N} \left[ \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2 \right]
\]
be the pooled estimator of the common variance $\sigma^2 = \text{Var} X_1 = \text{Var} Y_1$.

(a) [3 points] Let

$$T_k = \frac{Y_n - \bar{X}_m - \theta_0}{\sigma \sqrt{\frac{1}{m} + \frac{1}{n}}}.$$  

Prove that $\sqrt{N}(T_k)$ is asymptotically normal as $k \to \infty$ under the null hypothesis.

**Solution:** Let $\mu$ be the mean of the $F$ distribution. Under $H_0$, the central limit theorem gives

$$\sqrt{m}(\bar{X}_m - \mu) \xrightarrow{d} N(0, \sigma^2) \quad \text{and} \quad \sqrt{n}(\bar{Y}_n - \mu - \theta_0) \xrightarrow{d} N(0, \sigma^2).$$

Therefore, Slutsky’s theorem gives

$$\sqrt{N}(\bar{X}_m - \mu) \xrightarrow{d} N\left(0, \frac{\sigma^2}{\rho(1-\rho)}\right) \quad \text{and} \quad \sqrt{N}(\bar{Y}_n - \mu - \theta_0) \xrightarrow{d} N\left(0, \frac{\sigma^2}{1-\rho}\right).$$

Because the $X$ sample and the $Y$ sample are independent, we may subtract these two relations, resulting in

$$\sqrt{N}(Y_n - \bar{X}_m - \theta_0) \xrightarrow{d} N\left(0, \frac{\sigma^2}{\rho(1-\rho)}\right)$$

because $\rho^{-1} + (1-\rho)^{-1} = [\rho(1-\rho)]^{-1}$. We conclude that $\sqrt{N}(T_k) \xrightarrow{d} N(0, 1)$.

(b) [3 points] Prove that $S_k \xrightarrow{P} \sigma$ as $k \to \infty$, then argue that $\sqrt{N}(T_k)$ has the same asymptotic distribution as

$$\frac{Y_n - \bar{X}_m - \theta_0}{S_k \sqrt{\frac{1}{m} + \frac{1}{n}}}.$$

You should not assume without proof that an expression like $\frac{1}{m} \sum_{i=1}^{m} (X_i - \bar{X}_m)^2$ is consistent for $\sigma^2$.

**Solution:** Since

$$\frac{1}{m} \sum_{i=1}^{m} (X_i - \bar{X}_m)^2 = \left(\frac{1}{m} \sum_{i=1}^{m} X_i^2\right) - (\bar{X}_m)^2,$$

and since $\frac{1}{m} \sum_{i=1}^{m} X_i^2 \xrightarrow{P} E(X_1^2)$ and $\bar{X}_m \xrightarrow{P} E(X_1)$ by the WLLN, we conclude that

$$\frac{1}{m} \sum_{i=1}^{m} (X_i - \bar{X}_m)^2 \xrightarrow{P} E(X_1^2) - (EX_1)^2 = \sigma^2.$$

The same argument shows that $\frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2 \xrightarrow{P} \sigma^2$ (and because the distribution of this statistic does not depend at all on $E(Y_i)$, the argument is still valid even if $E(Y_i)$
changes as in part (c)). We conclude because \( m/N \to \rho \) and \( n/N \to 1 - \rho \) that \( S_k^2 \overset{P}{\to} \sigma^2 \).

Since
\[
\sqrt{N}(T_k) = \left[ \frac{\sigma \sqrt{N} \sqrt{\frac{N}{mn}}}{S_k \sqrt{\rho(1 - \rho)}} \right] \frac{Y_n - \bar{X}_m - \theta_0}{S_k \sqrt{\frac{1}{m} + \frac{1}{n}}},
\]
and the quantity in square brackets converges in probability to 1, this proves the desired result.

(c) [3 points] For \( k = 1, 2, \ldots \), let \( \theta_k = \theta_0 + C/\sqrt{N} \) for some positive constant \( C \). Suppose in parts (a) and (b) that we replace the \( Y_1, \ldots, Y_m \) by \( Y_{k1}, \ldots, Y_{km} \), assumed to be i.i.d. from \( F(t - \theta_k) \) for each \( k \).

Prove that \( \sqrt{N}[T_k - E(T_k)] \) is asymptotically normal under these assumptions.

Solution: The entire solution to part (a) still works as long as we can still verify that
\[
\sqrt{n}(Y_k - EY_{k1}) \overset{d}{\to} N(0, \sigma^2).
\]
To show this, we must merely check the Lyapunov condition because \( \text{Var} \ Y_{k1} \) is still \( \sigma^2 \). We obtain
\[
\frac{1}{n^2\sigma^4} \sum_{i=1}^{n} E(Y_{ki} - EY_{k1})^4 = \frac{E(Y_1 - EY_{k1})^4}{n\sigma^4},
\]
which must tend to zero as \( k \to \infty \) because \( n \to \infty \) as \( k \to \infty \). This verifies the Lyapunov condition and so asymptotic normality follows.

(d) [2 points] Based on part (a), we may reject the null hypothesis whenever \( T_k \) is larger than some cutoff (which may depend on \( k \)). The Wilcoxon rank-sum test is an alternative test of the same null hypotheses with efficacy equal to \( \sqrt{12\rho(1 - \rho)} \int_{-\infty}^{\infty} f^2(z) \, dz \). Find the asymptotic relative efficiency of the test in part (a) relative to the Wilcoxon rank-sum test. Evaluate this A.R.E. when \( f(z) \) is a normal density, in which case
\[
\int_{-\infty}^{\infty} f^2(z) \, dz = \frac{1}{2\sigma\sqrt{\pi}}.
\]

Solution: When \( \theta \) is the true value, we have
\[
E(T_k) = \frac{\sqrt{\rho(1 - \rho)}}{\sigma}(\mu + \theta - \mu) \quad \text{and} \quad \text{Var} \,(T_k) = \left( \frac{1}{n} + \frac{1}{m} \right) \rho(1 - \rho).
\]
In the notation of Chapter 8, this means that \( \mu(\theta) = \theta \sqrt{\rho(1 - \rho)} / \sigma \) and \( \tau^2(\theta_0) = \rho(1 - \rho)(N^2/mn) \). This latter expression has a limit of one as \( k \to \infty \). Therefore, the efficacy equals \( \sqrt{\rho(1 - \rho)} / \sigma \). We conclude that the A.R.E. equals
\[
\frac{1}{12\sigma^2(\int_{-\infty}^{\infty} f^2(z) \, dz)^2}.
\]
In the normal case, this is \( \pi/3 \).