Mar. 28 Announcements

- HW #8 is due on Friday, March 30 at 2:30pm.
- All homework must be turned in electronically from now on.
- See sample code for problems 1(a) and 1(b).

Mar. 28 Monte Carlo methods

How do you generate $X \sim F$ given $U \sim \text{Unif}(0, 1)$?

- Some special cases are covered in Ross, section 11.3
- General “inversion” method based on quantile function

$$F^{-1}(u) \overset{\text{def}}{=} \inf\{x : u \leq F(x)\}.$$  

Can prove: $F^{-1}(U) \sim F$, i.e., $P[F^{-1}(U) \leq x] = F(x)$ for all $x$.

**Notes:** In the special case when $F^{-1}(u)$, the inverse function, exists, there is a simple one-line proof of the fact above:

$$P[F^{-1}(U) \leq x] = P[F(F^{-1}(U)) \leq F(x)] = P[U \leq F(x)] = F(x).$$

Mar. 28 Monte Carlo methods

**Examples:**

- Generate $X \sim \text{Exponential}(1)$.
- Generate $X \sim \text{Standard Cauchy}$.

**Notes:** If $U$ is standard uniform, then the inversion method shows that $-\log(1 - U)$ is standard exponential and $\tan[\pi(U - \frac{1}{2})]$ is standard Cauchy.

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**Problems with the inversion method:**

- Only works in one dimension
- Explicit quantile functions are not always available
- Explicit $F(x)$ may not even be possible; for instance, $f(x)$ may be known only up to a constant.
Rejection sampling works much more generally than inversion method.
- Given: An “easier” density $g$ and known $K$ such that $Kg(x) \geq f(x)$.
- Idea: Sample repeatedly from $g$, but only keep each sampled value with probability $f(x)/Kg(x)$.

Notes: There is a nice graphical intuition here, which we discussed in class. The algorithm works like this:
1. Generate $X \sim g$.
2. Generate $U \sim \text{uniform}(0, 1)$.
3. If $U < f(X)/[Kg(X)]$, accept $X$; otherwise, reject it and start over.
4. Conditional on an acceptance, $X \sim f$.

Rejection sampling notes
- Only necessary to know $f$ and $g$ up to constants.
- Often better to take logarithms when computing with ratios.
- Section 11.2.2 has a couple examples (e.g., beta).

Notes: In step 3 in previous slide’s notes, it’s often easier (and safer numerically) to check whether

$$
\log U < \log f(X) - \log K - \log g(X).
$$

Suppose that
- $f(x) = \alpha r(x)$ and $g(x) = \beta s(x)$ are density functions
- $r(x)$ and $s(x)$ are known functions (but $\alpha$ and $\beta$ might be unknown).
- $K = \sup_x r(x)/s(x)$ is finite and known.
- Let $X \sim g(x)$ and $U \sim \text{unif}(0, 1)$ be independent.

Then
$$
X \mid U \leq \frac{r(X)}{Ks(X)} \sim f.
$$

Notes: The proof begins by writing $P[X \leq x \mid U \leq \frac{r(X)}{Ks(X)}]$ using the definition of conditional probability. Next, use the conditioning technique by conditioning on $X$ in both the numerator and the denominator, then simplify. Try it yourself!