For Wednesday, work old homework problems and new homework problems.

- Midterm exam: 7:00pm on Wednesday, Feb. 29 in 105 Willard. (NB: Change of room!)
- HW #6 will be due on Friday (Mar. 2) at 2:30.

Let $X$ be an exponential random variable. Without any computations, which of the following is correct?

- $E[X^2 | X > 1] = E[(X + 1)^2]$
- $E[X^2 | X > 1] = E[X^2] + 1$
- $E[X^2 | X > 1] = (1 + E[X])^2$
- None of these

Notes: Answer: A, because $X | X > 1$ has the same distribution as $X + 1$ (this is the memoryless property).

A doctor has scheduled two appointments, one at 1:00 and one at 1:30. The amounts of time that appointments last are independent exponential random variables with mean 30 minutes. If both patients are on time, the expected amount of time that the 1:30 appointment spends at the doctor’s office is...

- Less than 30 minutes
- Between 30 and 60 minutes
- More than 60 minutes

Notes: Answer: B. In the case that the first patient is done by 1:30, the expected time equals 30 minutes. In the case that the first patient is not done by 1:30, the expected time equals 60 minutes (30 minutes for the first patient and 30 minutes for the second). Therefore, the final answer is a weighted average of 30 and 60. More precisely, the weight on 30 minutes is the probability that the first patient will be finished by 1:30, which equals $1 - e^{-1}$.

In a two-server queueing system, customers arrive according to a Poisson process with rate $\lambda$. If at least one of the two servers is free when a customer arrives, the customer is immediately served; otherwise, the customer departs immediately and is lost. Service times are exponential (independently) with rate $\mu$. 
In the two-server system (customer rate=$\lambda$, service rate=$\mu$),

- Starting from both servers busy, what is the expected time until the next customer is served?
- Starting empty, find the expected time until both servers are busy.
- What is the expected time between two successive lost customers?

Notes: Answers (we discussed the first two in class; try getting the third on your own—and check my algebra!):

- $\frac{1}{2\mu} + \frac{1}{\lambda}$
- $(2\lambda + \mu)/\lambda^2$
- $\frac{1}{\lambda} + \frac{2\mu}{\lambda(\lambda + \mu)} + \frac{2\mu^2(2\lambda + \mu)}{\lambda^3(\lambda + \mu)}$

Consider a single server queueing system where customers arrive according to a Poisson process with rate $\lambda$, service times are exponential with rate $\mu$, and customers are served in the order of their arrival. If a customer arrives and finds $n-1$ others in the system, and $X$ is the number in the system when that customer departs, find the pmf of $X$.

Notes: Consider the customer-arrival process and the customer-servicing process, which are two Poisson processes with rates $\lambda$ and $\mu$, respectively. The sum of these processes is Poisson with rate $\lambda + \mu$, and each event in the summed process is an arrival with probability $\lambda/(\lambda + \mu)$ and a service-finish with probability $\mu/(\lambda + \mu)$. If you are the $n$th customer, then you will leave exactly at the $n$th service-finish. Thus, the question is how many people arrived (call this a success) before the $n$th service-finish (call this a failure) in repeated Bernoulli trials. This means that the total number of trials is negative binomial random variable, which leads to the pmf.