Feb. 8
Announcements

- Finish Reading Section 5.2 for Monday.
- No class on Friday.
- No office hours tomorrow (but next Thursday is back to normal)

Feb. 8
4.4 Limiting probabilities

- A word about limiting distribution vs. stationary distribution...
- Although ergodicity lists positive recurrence as a condition, it’s often more practical to
  - check irreducibility
  - check aperiodicity
  - solve $\pi^T = \pi^T P$ to get $\pi$ (if possible).

Notes: This slide is a comment on some recent questions I’ve gotten regarding the reading (Section 4.4) and the most recent homework. I mentioned that the limiting distribution $\pi$ satisfies

\[
\lim_{n \to \infty} P^n = \begin{bmatrix}
\pi^T \\
\pi^T \\
\vdots \\
\pi^T
\end{bmatrix}
\]

and the stationary distribution $\pi$ satisfies $\pi^T = \pi^T P$.

Feb. 8
4.8 Time Reversible Markov Chains

- Define $Q_{ij} = P(X_t = j \mid X_{t+1} = i)$.
- Use Bayes’ theorem to prove

\[
Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}
\]

- What if $P_{ij} = Q_{ij}$ for all $i, j$?

Notes: The proof of $Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$ is straightforward using the definition of conditional probability.

Feb. 8
4.8 Time Reversible Markov Chains

“Consider a stationary ergodic Markov chain (that is, an ergodic Markov chain that has been in operation for a long time)…”

- We now know what “ergodic” means:
- There is a limiting distribution $\pi$ that uniquely solves the equations $\pi^T = \pi^T P$.
- When \( P_{ij} = Q_{ij} \) for all \( i, j \), the process is said to be time reversible.
- To gain intuition, write
  \[
  Q_{ij} = P_{ij} = \frac{\pi_j P_{ji}}{\pi_i} \quad \text{as} \quad \pi_i P_{ij} = \pi_j P_{ji}.
  \]
- Interpretation: Proportionally, there are just as many \( i \to j \) transitions as \( j \to i \) transitions.

Notes: We had a lengthy discussion about the intuition; it is based on the idea that

\[
\begin{align*}
  P(\text{a randomly chosen transition is from } i \to j) &= \pi_i P_{ij}, \\
  P(\text{a randomly chosen transition is from } i \to j) &= \pi_j P_{ji}.
\end{align*}
\]

- Example of a time reversible Markov chain: random walk on \( 0, 1, \ldots, M \) where you can stay at the edges (0 and \( M \)) but you can’t go beyond them.
- The fact that this MC is time reversible can help find its stationary probabilities.

Notes: By inspection, we conclude that this MC is irreducible, aperiodic, and time reversible. This last condition gives us an additional tool to use in finding the limiting \( \pi \) vector. For instance, we argued that

\[
\alpha_0 = P_{01} = \frac{\pi_1}{\pi_0} P_{10} = \frac{\pi_1}{\pi_0} (1 - \alpha_1),
\]

where \( \alpha_0 \) and \( \alpha_1 \) are given. A whole series of these equations is derived in the “Consider a random walk with states 0, 1, \ldots, M \ldots” example in Section 4.8. As a special case, we ended class by defining the Ehrenfest model.