Feb. 6 4.4 Limiting Probabilities

Corollary: For an irreducible ergodic (time-homogeneous) Markov chain with transition matrix $P$, the equations

$$
\pi_j = \sum_{i=0}^{n} \pi_i P_{ij}, \quad j = 0, 1, 2, \ldots
$$

have solution

$$
\pi_j = \lim_{n\to\infty} P^n_{ij}.
$$

Notes: In Section 4.4, this “corollary” is part of the main theorem. We sketched a proof, which is exactly the same proof found in the textbook.

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Example:

- Let $\alpha$ be the probability of rain after a rainy day.
- Let $\beta$ be the probability of rain after a non-rainy day.

What are the limiting probabilities?

Notes: We argued that this chain must be irreducible and ergodic. Therefore the equations of the corollary give the limiting probabilities. These equations are

$$
\begin{align*}
\pi_1 &= \alpha \pi_1 + \beta \pi_2 \\
\pi_2 &= (1 - \alpha) \pi_1 + (1 - \beta) \pi_2 \\
\pi_1 + \pi_2 &= 1.
\end{align*}
$$
One last bit of terminology:
- stationary probability

Notes: The stationary probabilities are by definition the long-run probabilities.

More facts we will not prove:
- If we assume $\sum j \pi_j = 1$, then the solution in the corollary is unique.
- The $\pi_j$ from the corollary are the stationary probabilities.

Notes: These facts are given in various places in Section 4.4.

More facts we will not prove:
- Aperiodicity is not necessary in order for the equations in the corollary to have a unique solution. However, for irreducible chains, positive recurrence is necessary.
- Stationary probabilities exist even for periodic irreducible chains.

Notes: These facts are given in various places in Section 4.4.

“Consider a stationary ergodic Markov chain (that is, an ergodic Markov chain that has been in operation for a long time)…”
- We now know what “ergodic” means:
  There is a limiting distribution $\pi$ that uniquely solves the equations $\pi^T = \pi^T P$. 

Notes: These facts are given in various places in Section 4.4.
Define \( Q_{ij} = P(X_t = j \mid X_{t+1} = i) \).

Use Bayes' theorem to prove

\[
Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}.
\]

What if \( P_{ij} = Q_{ij} \) for all \( i, j \)?

Notes: We'll prove the fact about \( Q_{ij} \) in the next class. When \( P_{ij} \) and \( Q_{ij} \) are the same, the Markov chain has a special property called time-reversibility.