Feb. 1 4.6 Mean Time Spent in Transient States

- For transient states $i$ and $j$, let $s_{ij}$ equal the expected number of time steps spent in $j$, given $X_0 = i$.
- Let $S = (s_{ij})$ be the matrix of $s_{ij}$ values.
- Last time, we showed that $S = I + P_T S$.
- Therefore, $S = (I - P_T)^{-1}$.

Notes: This is review from Monday’s class.

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In our example,

$$P_T = \begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 0
\end{bmatrix}$$

and so

$$S = (I - P_T)^{-1} = \frac{1}{5} \begin{bmatrix}
8 & 6 & 4 & 2 \\
6 & 12 & 8 & 4 \\
4 & 8 & 12 & 6 \\
2 & 4 & 6 & 8
\end{bmatrix}.$$

Notes: Here, the theory developed so far is applied to the specific problem at hand (i.e., gambler’s ruin with $N = 5$, $i = 2$, and $p = 1/2$). Important note: I did not reorder the states, as is done in Section 4.6, so that all of the transient ones come first. Instead, this $P_T$ matrix is the $4 \times 4$ submatrix in the middle of the $6 \times 6$ full $P$ matrix, since the 4 transient states are all except for the first state (zero) and the last state (five).
In our example,

\[
S = \begin{bmatrix}
8 & 6 & 4 & 2 \\
6 & 12 & 8 & 4 \\
4 & 8 & 12 & 6 \\
2 & 4 & 6 & 8 \\
\end{bmatrix}
\]

Starting with $2$...
- Mean # of steps with $2$: $\frac{12}{5}$
- Mean # of steps before end: 6

Notes: These answers may be read directly from the $S$ matrix. For the second one, it's necessary to sum the entire second row because

\[
E(\text{time spent in all transient states}) = \sum_{i \text{ transient}} E(\text{time spent in state } i).
\]

Notes: The key is to argue that

\[
E(\# \text{ steps in } j \mid X_0 = i, X_t = j \text{ for some } t > 0) = \begin{cases} 
 s_{jj} + 1 & \text{if } i = j \\
 s_{jj} & \text{if } i \neq j 
\end{cases}
\]

and

\[
E(\# \text{ steps in } j \mid X_0 = i, X_t \text{ is never } j \text{ for any } t > 0) = \begin{cases} 
 1 & \text{if } i = j \\
 0 & \text{if } i \neq j 
\end{cases}
\]

This leads to $s_{ij} = [s_{jj} + I\{i = j\}] f_{ij} + I\{i = j\}(1 - f_{ij})$ using conditioning, as desired. Solving gives $f_{ij} = [s_{ij} - I\{i = j\}] / s_{jj}$.

Let $f_{ij} \overset{\text{def}}{=} P(X_t = j \text{ for some } t > 0 \mid X_0 = t)$
- Demonstrate that $s_{ij} = I\{i = j\} + f_{ij} s_{jj}$.
- Solve to find $f_{ij}$.

Notes: The key is to argue that

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Particular type of Markov chain
- $X_t = \text{size of } t^{th} \text{ generation, } t = 0, 1, 2, \ldots$
- Assume each individual produces offspring independently according to some distribution $P = \{P_0, P_1, P_2, \ldots\}$ on the nonnegative integers.
For a branching process \( X_0, X_1, X_2, \ldots \),
- Possible to demonstrate \( EX_n = \mu^n \)
- Possible to demonstrate \( \text{Var} X_n = \begin{cases} \sigma^2 \mu^{n-1} \left( \frac{1-\mu}{1-\mu} \right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases} \)

\[ \sigma^2 \] where \( \mu \) and \( \sigma^2 \) are the mean and variance of \( P \).

**Notes:** The argument used for this derivation is simple conditioning on \( X_{n-1} \), then recursively on \( X_{n-2}, X_{n-3}, \ldots \).

Generally, the most interesting cases involve \( P_0 > 0 \). In this case...
- what are the classes of states?
- which are recurrent / transient?
- what does this mean for the Markov chain in the long term?

**Notes:** We argued that when \( P_0 > 0 \), all states other than zero form a single transient class. Zero is its own (recurrent) class, since it is an absorbing state. We conclude that either the process grows without bound, or it dies out by eventually hitting zero.

A commonly asked question is: What is
\[ P(\text{population will die out} \mid X_0 = 1) ? \]

Call this probability \( \pi_0 \) (as in book). Then
\[ \pi_0 = \lim_{n \to \infty} P(X_n = 0 \mid X_0 = 1). \]

- Well-known fact: If \( \mu < 1 \), then \( \pi_0 = 1 \).
- Can you derive the equation \( \pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j \)?

**Notes:** In an epidemiological context, \( \mu \) is often called \( R_0 \). In a simple branching-process model, this means that \( R_0 = 1 \) is the threshold between an epidemic possibly occurring (\( X_n \) gets large with positive probability) and no epidemic possible (\( X_n \to 0 \) with probability one).