1. Consider a Markov chain on $\Omega = \{1, 2, 3, 4, 5, 6\}$ specified by the transition probability matrix
\[
P = \begin{bmatrix}
\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\
\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
(a) What are the (communicating) classes of this Markov chain? Is the Markov chain irreducible?
(b) Which states are transient and which are recurrent? Justify your answers.
(c) What is the period of each state of this Markov chain? Is the Markov chain aperiodic?
(d) Let $X_0$ be the initial state with distribution $\pi_0 = (0, \frac{1}{4}, \frac{3}{4}, 0, 0, 0)\top$ corresponding to the probability of being in states 1, 2, 3, 4, 5, 6 respectively. Let $X_0, X_1, X_2, \ldots$ be the Markov chain constructed using $P$ above. What is $E(X_1)$?
(e) What is $\text{Var}(X_1)$?
(f) What is $E(X_3)$?

2. Suppose that the probability of rain today depends on weather conditions from the previous three days. If it has rained for the past three days, then it will rain today with probability 0.7; if it did not rain for any of the past three days, then it will rain today with probability 0.1; if it rained each of the past two days but not three days ago, it will rain with probability 0.8; and, in any other case, the weather today will match yesterday’s weather with probability 0.6.
(a) Describe this process using a Markov chain, i.e., define a state space and the corresponding transition probability matrix for the process.
(b) Suppose you know that it rained on days one, two, and three. What is the probability that it will rain on day seven? (You are welcome to use a computer for this, but please explain what you did.)

3. Prove that if state $i$ is recurrent and state $i$ does not communicate with state $j$, then $P_{ij} = 0$. This implies that once a process enters a recurrent class of states, it can never leave that class. For this reason, a recurrent class is often referred to as a closed class.

4. Computer problem: Use the Markov chain described in Problem 1 and the initial distribution in 1d.
(a) Simulate a realization of the random variable $X_3$. Repeat this 1000 times—i.e., generate 1000 instances of $X_3$—and calculate the average. This is your estimate of $E(X_3)$. (Ideally, you should report some sort of confidence interval, but this is not required.) Compare your estimate with your answer from Problem 1. Since this is a short program, include a printout of your code with your homework.
(b) Simulate the Markov chain according to Problem 1 and run it for 100,000 steps. Now calculate the proportion of times the Markov chain was in the states 1, 2, 3, 4, 5, 6 respectively. Simulate two more realizations, each also of length 100,000, and again record the proportion of times the Markov chain was in the states 1, 2, 3, 4, 5, 6 respectively. You only have to report the proportions for each of the three realizations (do not print out your Markov chains or your code for this!)
(c) Use your answer to part (b) to find one or more vectors $v$ such that $v^\top P = v^\top$. Explain your reasoning.