**Question 1:** Suppose that $X_1, \ldots, X_n$ are a simple random sample from a Weibull distribution with density function

$$f_\theta(x) = \theta c x^{c-1} \exp\{-\theta x^c\} I\{x > 0\}$$

for some fixed and known value of $c$. (This situation comes up occasionally in the context of lifetime testing of mechanical equipment. When $c = 1$, a Weibull distribution is simply an exponential distribution.)

(a) Show that $X_i^c$ is exponentially distributed.

(b) If we wish to test $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$, derive the value $k_\alpha$ such that the test that rejects $H_0$ whenever $\sum_{i=1}^n X_i^c > k_\alpha$ is of size $\alpha$. Relate $k_\alpha$ to a critical value of the $\chi^2$ distribution on $2n$ degrees of freedom.

(c) Show that the test in part (b) is the U.M.P. level $\alpha$ test.

(d) Using a normal approximation to the distribution of the test statistic in part (b)*, give an explicit approximate form of the power function of the test. Then, use this approximation to find the sample size needed for a size 0.05 test to have power at least 0.9 at the alternative $\theta = 0.2$ if $\theta_0 = 0.25$.

*In other words, assume that the test statistic is approximately normally distributed with exactly the same mean and variance that it has in reality.

**Solution:**

(a) The Weibull random variable $X$ has a closed-form cumulative distribution function (cdf), given by

$$F_\theta(x) = 1 - \exp\{-\theta x^c\} I\{x > 0\}.$$ 

Therefore, the cdf of $X^c$ is

$$P(X^c < x) = P(X < x^{1/c}) = F_\theta(x^{1/c}) = 1 - \exp\{-\theta (x^{1/c})^c\} = 1 - \exp\{-\theta x\}$$

for $x > 0$, which is easily seen to be the cdf of an exponential distribution.

(b) Using the same logic as in part (a), we find that $\theta X^c$ has a standard exponential distribution:

$$P(\theta X^c < x) = P(X < (x/\theta)^{1/c}) = F_\theta((x/\theta)^{1/c}) = 1 - \exp\{-\theta (x/\theta)^{1/c}\} = 1 - \exp\{-x\}.$$ 

If $T = \sum_{i=1}^n X_i^c$, this means that $2\theta T$ has the same distribution as the sum of $n$ independent exponential random variables with mean 2. Since $\chi^2_2$ is the same as exponential with mean 2, we conclude that when $\theta = \theta_0$, $2\theta T$ is distributed as a chi-squared random variable with $2n$ degrees of freedom. This means that if we set

$$k_\alpha = \frac{\chi^2_{2n}(1 - \alpha)}{2\theta_0},$$

where $\chi^2_{2n}(1 - \alpha)$ is the $1 - \alpha$ quantile of the $\chi^2_{2n}$ distribution, then the test that rejects $H_0 : \theta \geq \theta_0$ whenever $T > k_\alpha$ is of size $\alpha$.

(c) The likelihood is

$$L(\theta) = (\theta c)^n \prod_{i=1}^n X_i^{c-1} \exp\{-\theta \sum_{i=1}^n X_i^c\} = (\theta c)^n \left[ \prod_{i=1}^n X_i^{c-1} \right] \exp\{-\theta T\}.$$ 

so $T$ is sufficient for this family of distributions. By the Karlin-Rubin Theorem, it suffices to verify that this family has a monotone likelihood ratio (MLR) in the statistic $T$. To verify this, we check that for $\theta_1 < \theta_2$,

$$\frac{L(\theta_2)}{L(\theta_1)} = \left( \frac{\theta_2}{\theta_1} \right)^n \exp\{(\theta_1 - \theta_2)T\}$$
is a strictly decreasing function of $T$.

(d) The result of part (a) shows that $X^c_i$ is exponential with mean $1/\theta$ and variance $1/\theta^2$. Therefore, $T$ has mean $n/\theta$ and variance $n/\theta^2$. We conclude that $(\theta T - n)/\sqrt{n}$ has mean 0 and variance 1. Using the normal approximation, with $Z$ denoting a standard normal random variable, the power function is

$$
\beta(\theta) = P_{\theta} (T > k_\alpha) = P_{\theta} \left( \frac{\theta T - n}{\sqrt{n}} > \frac{\theta k_\alpha - n}{\sqrt{n}} \right) \approx P_{\theta} \left( Z > \frac{\theta k_\alpha - n}{\sqrt{n}} \right) = 1 - \Phi \left( \frac{\theta k_\alpha - n}{\sqrt{n}} \right).
$$

It is not easy to find a closed-form solution for the smallest $n$ that gives $\beta(0.2) \geq 0.9$ when $\theta_0 = 0.25$, since $k_\alpha$ depends on $n$ in a complicated way. Thus, we’ll find this $n$ numerically. We can depict the power of the size 0.05 test at $\theta = 0.2$ for various values of $n$ as follows:

> n <- 1:200
> k <- qchisq(.95, 2*n) / (2*0.25) # This gives size 0.05 when theta0=0.25
> beta <- 1-pnorm((0.2*k-n)/sqrt(n))
> plot(n, beta, type="l")
> abline(h=0.9, lty=2) # Put a line at 0.9, the desired power

To find the smallest $n$ that achieves the desired power, we may use the `which` function with the `min` function:

> min(which(beta >= 0.9))

[1] 174

Many people used the normal approximation to find the value of $k_\alpha$ on which $\beta(0.2)$ is based—in fact, this might be the most sensible course of action given that I was not specific about whether to do this. Here is the result:

> k2 <- qnorm(.95, mean=n/0.25, sd=sqrt(n)/0.25) # Use normal approx for k_alpha
> beta2 <- 1-pnorm((0.2*k2-n)/sqrt(n))
> min(which(beta2 >= 0.9))
In fact, for the sake of comparison we can simply find the exact result, which does not use any normal approximation:

\[
\beta_3 \leftarrow \text{pchiq}(2 \times 0.2 \times k, 2 \times n)
\]

\[
\min(\text{which}(\beta_3 \geq 0.9))
\]

[1] 169

**Question 2:** Suppose that \(X_1, \ldots, X_m\) and \(Y_1, \ldots, Y_n\) are independent simple random samples, with \(X_i \sim N(\theta_1, \sigma^2)\) and \(Y_j \sim N(\theta_2, \sigma^2)\) for an unknown \(\sigma^2\). We wish to test \(H_0 : \theta_1 \leq \theta_2\) against \(H_1 : \theta_1 > \theta_2\). Show that the likelihood ratio test is equivalent to the two-sample \(t\)-test.

**Solution:** The likelihood function is

\[
L(\theta_1, \theta_2, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{(m+n)/2}} \exp\left\{ -\frac{1}{2\sigma^2} \left[ \sum_{i=1}^m (X_i - \theta_1)^2 + \sum_{j=1}^n (Y_j - \theta_2)^2 \right] \right\}.
\]

Maximizing with no constraints to obtain the denominator of the LRT statistic, we easily obtain \(\hat{\theta}_1 = \bar{X}\) and \(\hat{\theta}_2 = \bar{Y}\). This leads to

\[
\hat{\sigma}^2 = \frac{1}{m+n} \left[ \sum_{i=1}^m (X_i - \bar{X})^2 + \sum_{j=1}^n (Y_j - \bar{Y})^2 \right],
\]

and we will refer to the expression in Equation (1) as the pooled sample variance, or \(\hat{\sigma}^2_{\text{pooled}}\).

As for the numerator of the LRT statistic, we first note that in the case \(\bar{X} \leq \bar{Y}\), the unconstrained maximizer described above lies in the null space \(\Theta_0\), and we conclude that the LRT statistic equals one whenever \(\bar{X} \leq \bar{Y}\). However, in the case \(\bar{X} > \bar{Y}\), we must carry out a constrained maximization and find the largest possible value of \(L(\theta_1, \theta_2, \sigma^2)\) such that \(\theta_1 \leq \theta_2\). We will prove that the maximizer in this case occurs on the boundary, i.e., we must have \(\hat{\theta}_1 = \hat{\theta}_2\).

One way to prove this is to notice that if there is a local maximizer on the interior of \(\Theta_0\), i.e., such that \(\hat{\theta}_1 < \hat{\theta}_2\), then the gradient vector must be zero at that local max since it is on the interior of the space. But the unique zero of the gradient function satisfies \(\theta_1 = \bar{X}\) and \(\theta_2 = \bar{Y}\). So there is no interior local maximum.

Another way to prove that we only need to consider \(\theta_1 = \theta_2\) is as follows: We will show directly that when \(\bar{X} > \bar{Y}\) and \(\theta_1 < \theta_2\), either

\[
L(\theta_1, \theta_1, \sigma^2) > L(\theta_1, \theta_2, \sigma^2) \quad \text{or} \quad L(\theta_2, \theta_2, \sigma^2) > L(\theta_1, \theta_2, \sigma^2),
\]

which means that we can eliminate any point \(\theta_1 < \theta_2\) from consideration in the maximization problem. To prove (2), we may verify after a bit of algebra that

\[
\log L(\theta_1, \theta_1, \sigma^2) - \log L(\theta_1, \theta_2, \sigma^2) = \frac{n}{2\sigma^2} (\theta_2 - \theta_1)(\theta_1 + \theta_2 - 2\bar{Y}),
\]

\[
\log L(\theta_2, \theta_2, \sigma^2) - \log L(\theta_1, \theta_2, \sigma^2) = \frac{m}{2\sigma^2} (\theta_2 - \theta_1)(2\bar{X} - (\theta_2 + \theta_1)).
\]

Since \(\bar{X} > \bar{Y}\), at least one of the two expressions in square brackets must be greater than zero no matter what the value of \(\theta_1 + \theta_2\) is. This proves the result.

Maximizing \(L(\theta, \theta, \sigma^2)\) is straightforward because it is the same likelihood we would obtain if \(X_1, \ldots, X_m, Y_1, \ldots, Y_n\) were a simple random sample of size \(m+n\) from \(N(\theta, \sigma^2)\). Therefore, we find

\[
\hat{\theta} = \frac{\sum_{i=1}^m X_i + \sum_{i=1}^n Y_i}{m+n} \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{m+n} \left[ \sum_{i=1}^m (X_i - \hat{\theta})^2 + \sum_{i=1}^n (Y_i - \hat{\theta})^2 \right],
\]
where we will denote the latter expression by $S_0^2$. By writing $(X_i - \hat{\theta}) = (X_i - \bar{X} + \bar{X} - \hat{\theta})$ and $(Y_i - \hat{\theta}) = (Y_i - \bar{Y} + \bar{Y} - \hat{\theta})$, we may simplify the numerator of $S_0^2$ as follows:

$$\sum_{i=1}^{m} (X_i - \hat{\theta})^2 + \sum_{i=1}^{n} (Y_i - \hat{\theta})^2 = \sum_{i=1}^{m} (X_i - \bar{X})^2 + \sum_{i=1}^{n} (Y_i - \bar{Y})^2 + \frac{mn}{m+n} (\bar{X} - \bar{Y})^2.$$

In the case $\bar{X} > \bar{Y}$, the LRT statistic is therefore

$$\left(\frac{S_{pooled}^2}{S_0^2}\right)^{(m+n)/2} = \left(\frac{S_{pooled}^2}{S_{pooled}^2 + mn(\bar{X} - \bar{Y})^2/(m+n)}\right)^{(m+n)/2},$$

where

$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{m+n}{m+n-2}} S_{pooled}^2 \sqrt{\frac{1}{m} + \frac{1}{n}}}$$

is the usual two-sample $T$ statistic and $k$ is a constant depending only on $m$ and $n$.

Thus, rejecting $H_0 : \theta_1 \leq \theta_2$ when the LRT statistic is too small is equivalent to rejecting $H_0 : \theta_1 \leq \theta_2$ when $\bar{X} > \bar{Y}$ and $T^2$ is too large, which proves that the LRT is equivalent to the usual two-sample $T$ test.

**Question 3:** In each of the cases, calculate a valid $p$-value:

(a) We observe 20 successes in 90 Bernoulli trials to test $H_0 : p \leq 1/6$ against $H_1 : p > 1/6$.

(b) We observe data 4, 3, 1, 4, 1 from a sample of $n = 5$ independent Poisson($\lambda$) random variables and we wish to test $H_0 : \lambda \leq 2$ against $H_1 : \lambda > 2$.

(c) For the test in part (b), imagine that you have not yet observed the data, which still consist of $n = 5$ observations from Poisson($\lambda$). Plot the power function for the most powerful level 0.05 test and give its exact values at $\lambda = 2$ and $\lambda = 3$. Is the size exactly 0.05?

**Solution:**

(a) Since more extreme values of $X$ according to $H_1$ are larger values, we need to consider the probability that $X \geq 20$. Since $P_p(X \geq 20)$ is an increasing function of $p$ for $X \sim \text{binom}(90, p)$ (we can verify this easily for $p \leq p_0 = 1/6$ since the pmf at every value $\geq 20$ is an increasing function of $p$ for $p < 1/6$),

$$p\text{-value} = \sup_{p \leq p_0} P_p(X \geq 20) = P_{p_0}(X \geq 20).$$

We may therefore find the $p$-value as follows:

```r
> 1-pbinom(19, 90, 1/6)
[1] 0.104315
```

(b) In this case we know that $T = \sum_{i=1}^{5} X_i$ is sufficient and that $T$ is distributed as Poisson with parameter $5\lambda$. Since the observed value of $T$ in this case is 13, we use the same logic as in part (a) to find the $p$-value as

```r
> 1-ppois(12, 10)
[1] 0.2084435
```

(c) Here, it is easy to verify that the Poisson has a MLR property in $T = \sum_{i=1}^{5} X_i$.

Thus, the test should reject $H_0 : \lambda \leq 2$ when $T \leq c$ for some $c$, and the smallest $c$ which gives a level 0.05 test is found below:

```r
> min(which(ppois(1:50, 10) > .95))
[1] 6
```
In other words, our UMP level 0.05 test rejects whenever $T \geq 15$. The power function is the probability that this happens as a function of the true $\lambda$, so we may plot this function for a range of lambda values as follows:

```r
> lambda <- seq(1, 4, len=200)
> plot(lambda, 1-ppois(14, 5*lambda), type="l")
> abline(h=0.05, v=2:3, lty=2)
```

Finally, here are the values of $\beta(\lambda)$ for $\lambda = 2$ and $\lambda = 3$:

```r
> 1-ppois(14, 5*c(2,3))
[1] 0.08345847 0.53434629
```

We see that the exact size of the test is actually 0.083, not 0.05.