Question 1: Prove that in a one-dimensional canonical exponential family, the complete and sufficient statistic achieves the Cramér-Rao lower bound. That is, if

\[ f_\eta(x) = h(x) \exp\{\eta T(x) - A(\eta)\}, \]

where \( \eta \) and \( T(x) \) are scalars, then \( T(X) \) achieves the Cramér-Rao lower bound.

Solution:

Let \( \tau(\eta) = ET(X) \). Then \( \tau(\eta) = A'(\eta) \) and \( \text{Var}T(X) = A''(\eta) \). Therefore, the lower bound is achieved if

\[ A''(\eta) = \left[ A''(\eta) \right]^2 \frac{1}{I(\eta)}, \]

so it remains only to show that \( I(\eta) = A''(\eta) \). But this follows from

\[ -\frac{\partial^2}{\partial \eta^2} \log f_\eta(X) = A''(\eta), \]

because this second derivative is not a function of \( X \) and therefore not random, so taking an expectation doesn’t change anything and we conclude that \( I(\eta) = A''(\eta) \).

Question 2: Suppose that \( X \) has density function

\[ f_\theta(x) = \frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)} I\{x > 0\}. \]

Find expressions for the mean and variance of \( \log X \).

Solution:

Rewriting, we obtain

\[ f_\theta(x) = \exp\{\theta \log x - A(\theta)\} h(x) \]

for \( A(\theta) = \log \Gamma(\theta) \) and \( h(x) = x^{-1}e^{-x}I\{x > 0\} \). This is a canonical exponential family with sufficient statistic \( \log X \), which means that

\[ E \log X = \frac{d}{d\theta} A(\theta) = \frac{\Gamma'(\theta)}{\Gamma(\theta)} \]

and

\[ \text{Var} \log X = \frac{d^2}{d\theta^2} A(\theta) = \left( \frac{\Gamma''(\theta)}{\Gamma(\theta)} \right) - \left( \frac{\Gamma'(\theta)}{\Gamma(\theta)} \right)^2 \]

Question 3: Suppose that \( X_1, \ldots, X_n \) \( \overset{iid}{\sim} \) Uniform(0, \( \theta \)). Define \( T(X) = \max_i X_i \). You should be able to derive the density function of \( T \) (Hint: First derive the cumulative distribution function of \( T \) by noticing that \( T \leq t \) if and only if \( X_i \leq t \) for each \( i \) and show that \( T \) is complete and sufficient. Refer to Example 6.2.23 if you can’t remember how to do this, but you may simply assume these facts for this question.

(a) Derive the UMVU estimators of \( \theta \) and \( \theta^2 \).

(b) \( 2\bar{X} \) is an unbiased estimator of \( \theta \). Taking \( n = 100 \), perform a simulation study using \( \theta = 5 \) that compares the UMVUE with \( 2\bar{X} \) in terms of mean squared error. Use at least a thousand repetitions in your study.

(c) The (unbiased version of the) sample variance, \( S^2 \), is an unbiased estimator of \( \theta^2/12 \). Conduct a simulation study similar to the one in part (d), also with \( n = 100 \) and \( \theta = 5 \), in which you compare the UMVUE of \( \theta^2 \) to \( 12S^2 \).

Solution:
Since \( P(T \leq t) = [P(X_i \leq t)]^n \), we conclude that \( T \) has cumulative distribution function \( F_θ(t) = (t/θ)^n \) and density function \( f_θ(t) = n t^{n-1}/θ^n \) for \( 0 < t < θ \).

(a) Since \( T \) is complete and sufficient, any unbiased estimator that is a function of \( T \) must be UMVU. We find that

\[
E_θ T = \frac{n}{θ^n} \int_0^θ t^n \, dt = \frac{nθ}{n+1}
\]

and

\[
E_θ T^2 = \frac{n}{θ^n} \int_0^θ t^{n+1} \, dt = \frac{nθ^2}{n+2},
\]

which implies that \((n+1)T/n\) and \((n+2)T^2/n\) are UMVU estimators of \( θ \) and \( θ^2 \), respectively.

(b) The following code generates a matrix of 1000 samples of size \( n = 100 \) from \( \text{uniform}(0,5) \):

\[
> n <- 100; \text{reps}=1000
> \text{set.seed}(321)
> m <- \text{matrix}(5*\text{runif}(n*\text{reps}), n, \text{reps})
\]

Now we apply both estimators to the samples, which are in the columns of the matrix:

\[
> \text{est1} <- (n+1)/n*\text{apply}(m, 2, \text{max})
> \text{est2} <- 2*\text{apply}(m, 2, \text{mean})
> \text{c}(\text{UMVU}=\text{mean}((\text{est1}-5)^2), \text{alternative}=\text{mean}((\text{est2}-5)^2))
\]

UMVU alternative
0.002392229 0.081572157

We see that the UMVU is much more precise than the sample mean-based estimator.

(c) We can use the same samples generated for part (b):

\[
> \text{est3} <- (n+2)/n*\text{apply}(m, 2, \text{max})^2
> \text{est4} <- 12*\text{apply}(m, 2, \text{var})
> \text{c}(\text{UMVU}=\text{mean}((\text{est3}-25)^2), \text{alternative}=\text{mean}((\text{est4}-25)^2))
\]

UMVU alternative
0.2345324 5.2532793

Again, the UMVU is a great deal more precise than the alternative, which in this case is based on the sample variance.

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**Question 4:** It is not necessarily a good idea to restrict oneself to considering only unbiased estimators.

(a) In the example of Question 3, find the value of \( α \) that minimizes the MSE of \( αT \) as an estimator of \( θ \). (Hint: The answer is not 1.)

(b) Suppose that \( X \) has a truncated Poisson(\( θ \)) distribution; that is,

\[
P_θ(X = x) = \frac{θ^x e^{-θ}}{x!(1 - e^{-θ})} I \{x \in \{1, 2, 3, \ldots\}\}.
\]

Define \( τ(θ) = e^{-θ} \), the probability mass at zero for the untruncated Poisson distribution. Find the UMVUE of \( τ(θ) \). (Hint: The correct answer should look pretty silly!)

(c) For the situation in part (b), conduct a simulation study in which you compare the UMVUE with some other estimator (say, the method of moments estimator, though you may use another estimator if you prefer) in terms of MSE. Take \( θ = 5 \).

Below is one way to create a function to generate from a truncated Poisson distribution (I saw this elegant method in a message board post by Peter Dalgaard). See if you can convince yourself that it works, though you don’t need to write that part up for your homework!
rtipois <- function(repetitions, theta)
    qpois(runif(repetitions, dpois(0, theta), 1), theta)

Solution:

(a) First, the “Hint” is not worded correctly, since the value $\alpha = 1$ would not make
the estimator $\alpha T$ unbiased. The hint should say that the value is not $(n+1)/n$. Ignoring
an additive constant that does not involve $\alpha$, the MSE is

$$\alpha^2 E\tau^2 - 2\alpha\theta E\tau = \theta^2 \left( \frac{\alpha^2}{n+2} - \alpha \frac{2n}{n+1} \right),$$

which is a quadratic function minimized at $\alpha = (n+2)/(n+1)$.

(b) If $W(X)$ is an unbiased estimator of $\tau(\theta)$, then

$$e^{-\theta} = E_\theta W(X) = \sum_{x=1}^{\infty} \frac{W(x)\theta^x e^{-\theta}}{x!(1-e^{-\theta})}.$$

Plugging in the power series for $e^{-\theta}$, this simplifies to

$$-\sum_{x=1}^{\infty} \frac{(-\theta)^x}{x!} = \sum_{x=1}^{\infty} \frac{W(x)\theta^x}{x!}.$$

Since this equation must be true for all $\theta$, the only possible estimator $W(X)$ equals $-(-1)^X$. In other words,

$$W(X) = \begin{cases} 1 & \text{if } X \text{ is odd;} \\ -1 & \text{if } X \text{ is even} \end{cases}$$

(c) We can obtain a MME by noting that $E(X) = \theta/(1-e^{-\theta})$. Writing $\tau = \exp^{-\theta}$, this may be reexpressed as $E(X) = -\log \tau/(1-\tau)$. (Remember, $\theta$ is positive, which
means that $0 < \tau < 1$.)

We have only a single observation $X$ in this problem. Thus, the MME, which we’ll
denote by $\hat{\tau}$, may be found by solving for $\tau$ in the equation

$$\frac{-\log \tau}{1-\tau} = X.$$

Incidentally, it is not hard to verify that the MLE is the solution to the same equation.
However, there are two problems with this equation: First, it has no solution for $\tau \in (0,1)$
when $X = 1$. So we’ll allow the possibility of $\hat{\tau} = 1$ (corresponding to $\hat{\theta} = 0$) in this
case. Second, there is no closed-form solution in general, so we’ll have to use a numerical
method to solve it.

Solving the equation above numerical is easy to do here via any number of methods.
In the simulation study below, I will use the \texttt{uniroot} function. First, we simulate the
data:

```r
> reps <- 1000; th <- 5; tau <- exp(-th)
> rtpois <- function(repetitions, theta)
+     qpois(runif(repetitions, dpois(0, theta), 1), theta)
> set.seed(123)
> x <- rtpois(reps, th)
```

Next, a function that will find the root (zero) of the function

$$h(\tau) = X(1-\tau) + \log \tau,$$

allowing for the special case when $X = 1$, is given as follows:

```r
> findTauHat <- function(x)
+     ifelse(x==1, 1,
+             uniroot(function(tau, data)
+                     data*(1-tau) + log(tau),
+                     data=x, lower=1e-10, upper=1-1e-10)$root
+     )
```
Now we can apply both estimators to the vector of simulated data:

```r
> unbiased <- 2*(x %% 2) - 1 # Here, (x %% 2) means x mod 2.
> mean((unbiased-exp(-5))^2)
[1] 1.000611
> alternative <- sapply(x, findTauHat)
> mean((alternative-exp(-5))^2)
[1] 0.03440323
```

Since $e^{-5}$ is very close to zero and the unbiased estimator is ±1, it is not surprising that its MSE is close to 1. On the other hand, the method of moments estimator has a much smaller MSE even though it is slightly biased.