Stat 514 Solution 16 F. Xuan.

7.3.5. (a) $T=\frac{1}{n}\sum X_i^2$ (b) $T=\frac{1}{n}\sum X_i$ (c) $T=\frac{n+1}{n}Y_n$

are unbiased for 7.2.1, 7.2.2, 7.2.3 respectively

7.4.5. $f(x_1,\cdots x_n;\theta)=e^{n\theta}\cdot e^{-\sum x_i}I(\theta<\min(x))$

$=(e^{-\sum x_i})\cdot e^{n\theta}I(\theta<\min(x))$

$\Rightarrow$ $Y_1=\min(x)$ is sufficient.

Notice $f_{Y_1}(y)=ne^{-n(y-\theta)}I(y>\theta)$

$\Rightarrow$ $E\,u(Y_1)=\int_\theta^{+\infty}u(y)\,n\cdot e^{-ny}\cdot e^{n\theta}dy=0$

$\Leftrightarrow$ $\int_\theta^{+\infty}u(y)e^{-ny}dy=0$

$\Leftrightarrow$ $u(y)=0$ $\forall y>0$ since above is a laplace transform of $u(y)$.

$\Rightarrow$ Y_1 is complete.

By the fact that $EY_1=\theta+\frac{1}{n}$ $\Rightarrow$ $Y_1-\frac{1}{n}$ is MVUE

7.5.2. $f(x_1,\cdots x_n;\theta)=\theta^n e^{-\theta\sum x_i}$

$\Rightarrow$ $Y=\sum X_i$ is sufficient

$f(x)=\theta\cdot e^{-\theta x}$ belongs to the exponential family

$\Rightarrow$ $\sum X_i$ is complete

Notice $Y\sim \text{Gamma}(n,\frac{1}{\theta})$

$E\frac{n-1}{Y}=n-1\cdot\int\frac{\theta^n}{(n-1)!}y^{n-2}e^{-\theta y}dy$

$=\theta\int\frac{\theta^{n-1}}{(n-2)!}y^{n-2}e^{-\theta y}dy=\theta$

$\Rightarrow$ $\frac{n-1}{Y}$ is MVUE

7.5.3. sufficient is obvious.

complete comes from the fact of exponential family

(b). $l = \Sigma \log f(x_i;\theta) = n\log\theta + (\theta-1)\Sigma\log X_i$

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \Sigma\log X_i$$

$$\Rightarrow \hat{\theta} = -\frac{n}{\Sigma\log X_i} = -\frac{1}{\log T}$$

7.5.10. sufficient is obvious.

exponential family $\Rightarrow$ complete

Notice $X \sim \text{Gamma}(2, \frac{1}{\theta})$

$$\Rightarrow Y = \Sigma X_i \sim \text{Gamma}(2n, \tfrac{1}{\theta})$$

$$\Rightarrow \frac{2n-1}{Y} \text{ is UMVUE} \quad (\text{see } 7.5.2).$$

7.5.13. $l(\theta) = (\Sigma X_i)\log\theta + n\log(1-\theta)$. Let $Y = \Sigma X_i$

$$\Rightarrow \frac{\partial l}{\partial\theta} = \frac{Y}{\theta} - \frac{n}{1-\theta} \Rightarrow \hat{\theta} = \frac{Y}{n+Y}.$$

(b). same as 7.5.10.

(c). $Y = \Sigma X_i \sim$ Negative binomial (n, p).

Consider $u(X_1) = \begin{cases} 1 & \text{if } X_1 = 0 \\ 0 & \text{e.w.} \end{cases} \Rightarrow Eu(x) = 1-\theta$

$\Rightarrow 1-u(x)$ is unbiased

Let $T = E(1-u(x) \mid Y)$ is MVUE

$$= 1 - P(X_1 = 0 \mid \Sigma X_i = y) = 1 - \frac{P(X_1=0, \Sigma X_i = y)}{P(\Sigma X_i = y)}.$$

$$= 1 - \frac{\binom{n+y-2}{y}}{\binom{n+y-1}{y}} = 1 - \frac{n-1}{y+n-1} = \frac{y}{y+n-1} = \frac{\Sigma X_i}{\Sigma X_i + n - 1}$$