On r-Quick Limit Sets for Empirical and Related Processes Based on Mixing Random Variables

GUTTI JOGESH BABU* AND KESAR SINGH

Indian Statistical Institute, Calcutta, India, and Rutgers University

Communicated by T. L. Lai

The r quick limit points of normalized sample paths and empirical distribution functions of mixing processes are characterized. An r-quick version of Bahadur–Kiefer-type representation for sample quantiles is established, which yields the r-quick limit points of quantile processes. These results are applied to linear functions of order statistics. Some results on r-quick convergence of certain Gaussian processes are also established.

1. INTRODUCTION

Let \( \{f_n\} \) be a sequence of measurable functions taking values in a metric space \((M, d)\). Let \( r > 0 \) and \( U \subset M \). The sequence \( \{f_n\} \) is said to be contained r-quickly in \( U \) if for each \( \epsilon > 0 \),

\[
E(\sup(n \geq 1: f_n \notin U_\epsilon))' < \infty,
\]

where \( U_\epsilon \) denotes the \( \epsilon \)-neighbourhood of \( U \). Supremum over an empty set will be taken to be zero. Following Strassen [17] and Lai [12], we say that \( \{f_n\} \) is r-quickly relatively compact in \( M \) if \( \{f_n\} \) is r-quickly contained in a relatively compact subset of \( M \). A point \( x \) of \( M \) is said to be an r-quick limit point of \( \{f_n\} \) if for each \( \epsilon > 0 \),

\[
E(\sup(n \geq 1: d(x, f_n) \leq \epsilon))' = \infty.
\]

The set of r-quick limit points of \( \{f_n\} \) is denoted by \( \hat{\partial}_r(f) \).

Received April 21, 1980; revised February 4, 1982.

AMS 1970 Subject Classification: 60F99, 60F05, 62L10.

Key words and phrases: r-quick limit points, r-quick convergence, Gaussian processes, Brownian motion, empirical process, quantile process, linear functions of order statistics, reproducing kernel Hilbert space, \( \phi \)-mixing, strong mixing.

* On leave from Indian Statistical Institute, Calcutta, India.

0047-259X/82/040508-18$02.00/0

Copyright © 1982 by Academic Press, Inc.
All rights of reproduction in any form reserved.
The main objective of the present paper is to characterize the \( r \)-quick limit points of properly normalized empirical and quantile processes based on mixing random variables. These results do not seem to be known even in the independent case.

The ideas of Chover [6] and Philipp [15] are used in proving the main results. The technique involves developing an \( r \)-quick version of Cramér–Wöld device and results similar to Theorem 2 of Lai [12], for mixing random variables (see Theorems 1, 2 and 3). The proof of Theorem 1 given in this paper differs from that of Lai’s and uses estimates of tail probabilities of sample means for mixing random variables. See Babu, Ghosh and Singh [3] and Babu and Singh [2].

The results on quantile processes are derived from the corresponding results for empirical processes and \( r \)-quick versions of Bahadur–Kiefer-type representations. These results are then applied to linear functions of order statistics. Some applications to the asymptotic distribution and moments of first exit times of linear functions of order statistics are given. Such exit times are useful in sequential analysis. The \( r \)-quick limit sets for certain Gaussian processes are characterized in the last section.

Throughout this paper, by a stationary process we mean a strictly stationary process.

In this paper, \( r \) denotes positive real number; \( p, m, n, j \) denote positive integers; \( k, i \) denote non-negative integers and \( q \) denotes an integer; \( T \) denotes a subset of \([0, 1]\) containing 0 and 1; \( t, s \) denote points in \([0, 1]\).

2. Preliminaries

In this section we prove a proposition which reduces our basic problem to the finite-dimensional case. This will help to simplify the presentation in the subsequent sections.

We start with some notations. For \( f \in D[0, 1], U \subset D[0, 1], \delta > 0 \) and \( T, \)
let \( f^T \) denote the restriction of \( f \) to \( T, \ U^T = \{ g^T: g \in U \}, \ |f^T|_\infty = \sup_{t \in T} |f^T(t)|, \ |f| = \sup_{t < s < 1} |f(t)|, \ \omega(\delta, f) = \sup_{|t-s| < \delta} |f(t) - f(s)|, \) and
\( U_\delta = \{ h \in D[0, 1]: \| h - g \| < \delta \) for some \( g \in U \} \).

Proposition 2.1. Let \( \{ f_n \} \) be a sequence of measurable functions taking values in \( D[0, 1]. \) Suppose \( L \) is compact in \( C[0, 1]. \) For each finite \( T, \) let \( \{ f_n^T \} \) be \( r \)-quickly contained in \( L^T \) and \( L^T \subset \partial_r(f^T). \) If for each \( \varepsilon > 0, \) there exists a \( \delta > 0 \) such that
\[
E(\sup(n \geq 1: \omega(\delta, f_n, \varepsilon))^r < \infty, \tag{2.1}
\]
then \( \{ f_n \} \) is \( r \)-quickly contained in \( L \) and \( \partial_r(f) = L. \)
Proof. Fix \( \varepsilon > 0 \), let \( \delta \) be as in (2.1). Then for any \( 0 < \delta' \leq \delta, \ g \in L \) and for any \( T = \{ t_i: \ 0 \leq i \leq m \}, \ 0 = t_0 < \cdots < t_m = 1 \) with \( \max_{1 \leq i \leq m} (t_i - t_{i-1}) \leq \delta' \),

\[
\| f_n - g \| \leq | f_n^T - g^T |_{\infty} + \omega(\delta', g) + \omega(\delta, f_n).
\] (2.2)

By the Arzelà-Ascoli theorem (see Billingsley [5]), there exists a \( 0 < \delta' \leq \delta \) such that \( \omega(\delta', g) < \varepsilon \) for all \( g \in L \). So by (2.1) and (2.2) it follows that \( \{f_n\} \) is \( r \)-quickly contained in \( L \) and hence \( \partial_r(f) \subset L \). To complete the proof, we have to show that \( L = \partial_r(f) \). This follows again from (2.1) and (2.2) since for any continuous function \( g \) on \([0, 1]\), \( \omega(\eta, g) \to 0 \) as \( \eta \to 0 \).

3. Sample Path Processes

Throughout this section let \( s_n = (2n \log n)^{1/2} \). Let \( \{X_n\} \) be a sequence of random variables. Let \( \sigma^2 = V(X_1) + 2 \sum_{n=2}^{\infty} \text{cov}(X_1, X_n) \), whenever this sum exists. Let \( S_0 = 0 \) and \( S_n = \sum_{j=1}^{n} X_j \) for \( n \geq 1 \). Define \( f_n \in D[0, 1] \) and \( g_n \in C[0, 1] \) by

\[
f_n(t) = s_n^{-1} S_{\lfloor nt \rfloor},
\] (3.1)

and \( g_n(t) = f_n(t) \) at \( t = i/n \), and \( g_n \) is linear on \([ (i - 1)/n, i/n ] \), \( i = 1, \ldots, n \). Clearly \( \| f_n - g_n \| \leq s_n^{-1} \max_{j \leq n} |X_j| \), and for any sequence \( \{Z_n\} \) of random variables and \( \rho > 1 \),

\[
E( \sup(k \geq 1: |Z_k| > \varepsilon) )^r \leq \sum_{n=1}^{\infty} \rho^{nr} P( \max_{\rho^{n-1} < k < \rho^n} |Z_k| > \varepsilon ).
\] (3.2)

If for some \( r > 0 \), \( \sup_n E |X_n|^{2 + 2r} < \infty \), then by (3.2) we have, for any \( \varepsilon > 0 \),

\[
E( \sup(n \geq 1: \max_{j \leq n} |X_j| > \varepsilon s_n) )^r
\]

\[
\leq \sum_{n=1}^{\infty} e^{(r+1)n} \max_{j \leq e^n} P( |X_j| > \varepsilon ((n-1)e^{n-1})^{1/2}) < \infty.
\]

It follows that \( \{f_n\} \) is \( r \)-quickly relatively compact in \( D[0, 1] \) if and only if \( \{g_n\} \) is so in \( C[0, 1] \) and that \( \partial_r(f) = \partial_r(g) \). So it is enough to study the \( r \)-quick limit behaviour of one of the processes \( \{g_n\} \) and \( \{f_n\} \). In this section we study \( \{f_n\} \) for stationary mixing processes. For definitions of \( \phi \)-mixing and strong mixing \((\alpha\text{-mixing})\) sequences see Deo [8].

For a sequence \( \{Z_n\} \) of random variables we write \( r \)-lim sup \( Z_n = z \) if \( E( \sup(n \geq 1: Z_n > c) )^r < \infty \) for \( c > z \) and \( = \infty \) for \( c < z \).
Let $\Phi$ denote the distribution function of the standard normal variable on the line and

$$K_r = \left\{ f \in C[0, 1], f(0) = 0, f \text{ is absolutely continuous} \right\}$$

and

$$\int_0^1 (f'(t))^2 \, dt \leq r.$$ 

It is well known that $K_r$ is compact in $C[0, 1]$.

**Theorem 1.** Let $\{X_n\}$ be a stationary $\phi$-mixing sequence of random variables with $E(X_i) = 0$ and $\phi(n) \leq n^{-\theta}$, where $\theta > \max(2, r)$. Suppose $E|X_i|^{2 + 2r} < \infty$ and $\sigma > 0$. Then $\{f_n\}$ defined by (3.1) is $r$-quickly contained in $K_{r\sigma^4}$ and $a_r(f) = K_{r\sigma^4}$.

**Remark 3.1.** It is well known that the infinite series defining $\sigma^2$ converges absolutely under the conditions of Theorem 1.

**Remark 3.2.** In the i.i.d. case, Lai [12] has shown this result under a weaker moment condition, namely, $E|X_i|^{2 + 2r}(1 + \log^+ |X_i|)^{-1-r} < \infty$. He has also shown that this moment restriction is necessary for the result to hold.

We divide the proof of Theorem 1 into several lemmas.

**Lemma 3.1.** Let $\{X_n\}$ be a sequence of random variables. Suppose for some $b > 1$ and $\eta > 1$

$$\max_{k \geq 1} P(\max_{l < n} |S_{l+k} - S_k| > b s_n) \leq n^{-r}(\log n)^{-\eta}. \quad (3.3)$$

Then for any $\epsilon > 0$, there exist $\delta > 0$ and $\rho > 1$ such that (2.1) and

$$P(\sup_{n < \rho \leq \rho_n} ||f_n - f_k|| > \epsilon) \leq n^{-r}(\log n)^{-\eta} \quad (3.4)$$

hold for $\{f_n\}$ defined by (3.1).

**Proof.** Observe that for any $\epsilon > 0$, there exists a positive $\delta < 1$ such that $s_n \epsilon > b s_{[n\delta]}$ and that for $t \in [0, 1 - \delta]$ and $t \leq s \leq t + \delta$,

$$s_n |f_n(s) - f_n(t)| \leq \max_{l \leq 1 + \delta} |S_{[nt]} + l - S_{[nt]}|. \quad (3.5)$$

So by (3.3) and (3.5) we have for $t \in [0, 1 - \delta]$

$$P(\sup_{t \leq s \leq t + \delta} |f_n(t) - f_n(s)| > \epsilon) \leq C(t) n^{-r}(\log n)^{-\eta}, \quad (3.6)$$
where \( C(t) \) is a constant depending on \( t \). Now for \( 1 < \rho < 2, n \leq m \leq \rho n, \)
\[
s_n \| f_n - f_m \| \leq \sup_i \| S_{1,i,t} + S_{1,m,t} \| + \sup_i \| S_{1,i,t} \| (1 - s_n/s_m)
\]
\[
\leq \max \left( \max_{j < 1 + n(p-1)} |S_{i+j} - S_i| \right) + \max \| S_i \| (1 - s_n^2/s_m^2)
\]
\[
\leq \frac{4}{4^{(p-1)}} \left( \max_{j < k < 4^{(p-1)}} \max_{k < k(n(p-1))} |S_i - S_{n(k-1)(p-1)}| \right)
\]
\[
+ \max \| S_i \| (\rho - 1 + ((\rho \log \rho)/\log n))
\]

So by (3.3), there exists a \( \rho > 1 \) such that
\[
P(\sup_{n < k < \rho n} \| f_n - f_k \| > \varepsilon) \ll n^{-r}(\log n)^{-n}. \tag{3.7}
\]

To prove (2.1), let \( 0 = t_0 < \cdots < t_m = 1 \) be such that \( \max_j (t_j - t_{j-1}) < \delta. \)
Since for \( n \leq k \leq \rho n, \)
\[
\omega(\delta, f_k) \leq 2 \| f_k - f_n \| + \omega(\delta, f_n)
\]
\[
\leq 2 \| f_k - f_n \| + 3 \max_{i < j < m} \left( \sup_{t_{j-1} < s \leq t_j} |f_n(t_{j-1}) - f_n(s)| \right),
\]
(2.1) follows from (3.2), (3.6) and (3.7). This completes the proof.

**Lemma 3.2.** Let \( \{Z_n\} \) be a sequence of \( \phi \)-mixing random variables. Let \( Y_n = \sum_{i=1}^{n} Z_i \) for \( n \geq 1 \) and \( Y_0 = 0. \) If for some \( a > 0, p > 1, \phi(p) < 1/4 \) and \( \max_{i \leq n} P(\| Y_n - Y_i \| > a) < 1/4, \) then for all \( n > p, \)
\[
P(\max_{i \leq n} \| Y_i \| > 3a) \leq 2P(\| Y_n \| > a) + 2n \max_{i \leq n} P(\| Z_i \| > a/p).
\]

**Proof.** Let \( E_j = (\max_{i < j} \| Y_i \| \leq 3a, \| Y_j \| > 3a) \) and \( E = \bigcup_{j=n}^{n} E_j. \) We have for \( n > p, \)
\[
P(E) \leq P(\| Y_n \| > a) + \sum_{j=1}^{n-1} P(E_j \cap (\| Y_n - Y_j \| > 2a))
\]
\[
\leq P(\| Y_n \| > a) + \sum_{j=1}^{n-p} P(E_j \cap (\| Y_n - Y_{j+p} \| > a))
\]
\[
+ \sum_{j=1}^{n-p} P(\| Y_{j+p} - Y_j \| > a) + \sum_{j=n-p+1}^{n} P(\| Y_n - Y_j \| > a)
\]

\[
\ll P(\| Y_n \| > a) + \sum_{j=1}^{n-p} P(\| Y_{j+p} - Y_j \| > a) + \sum_{j=n-p+1}^{n} P(\| Y_n - Y_j \| > a)
\]
\[ r - \text{QUICK LIMIT SETS} \]

\[ P(\{ Y_n > a \} + \sum_{j=1}^{n-p} P(E_j) | P(\{ Y_n - Y_{j+p} > a \} + \phi(p)) + np \max_{j<n} P(\{ Z_j > a/p \}) \]

\[ \leq P(\{ Y_n > a \} + \frac{1}{2} \sum_{j=1}^{n} P(E_j) + np \max_{j<n} P(\{ Z_j > a/p \}). \]

Since \( P(E) = \sum_{j=1}^{n} P(E_j) \), the result follows.

**Remark 3.3.** If \( \{ Z_n \} \) above is \( \alpha \)-mixing, then for \( 1 \leq j \leq n-p \)

\[ P(E_j \cap (\{ Y_n - Y_{j+p} > a \}) \leq P(E_j) P(\{ Y_n - Y_{j+p} > a \}) + \alpha(p) \]

\[ \leq \frac{1}{2} P(E_j) + \alpha(p). \]

So in this case we have, for \( 1 < p < n \), that

\[ P(\max_{i<n} | Y_i > 3a) \leq 2P(\{ Y_n > a \} + 2n\alpha(p) + 2np \max_{j<n} P(\{ Z_j > a/p \}). \]

**Lemma 3.3** (See Theorem 5 and Remark 4.1 of Babu, Ghosh and Singh [3].) Under the conditions of Theorem 1, there exists a \( \lambda > 0 \) such that for all \( u^2 < (1 + 2r) \log n \),

\[ P(S_n > \sigma u_n^{1/2} - \Phi(-u) \ll n^{-\lambda} e^{-u^2/2} + n^{-r}(1 + |u|)^{-2-2r}. \]

**Lemma 3.4.** Let \( 0 = t_0 < t_1 < \cdots < t_p = 1, p \geq 2 \). Under the conditions of Theorem 1 we have, for any \( 0 < \varepsilon < 1 \), that

\[ P \left( \sum_{i=1}^{p} \left( f_n(t_i) - f_n(t_{i-1}) \right)^2 / (t_i - t_{i-1}) > (1 + 3\varepsilon) r\sigma^2 \right) \ll n^{-r}(\log n)^{-1-r}. \]

**Proof.** We shall prove the result for the case \( p = 2 \). The proof for \( p \geq 2 \) is similar. For notational convenience we put \( t = t_1, s = 1 - t, k = [nt], b_n = 2(1 + \varepsilon) r\sigma^2 n \log n, m = 2 + [1/\varepsilon] \) and \( i = k - [nt^2 \varepsilon^2/8] \). We have, by the stationarity of the sequence \( \{ X_n \} \) and by Lemma 3.3, that

\[ P(s^{-1}(S_n - S_k)^2 + t^{-1}S_t^2 > b_n) \]

\[ \leq P(m(S_n - S_k)^2 > s(m - 1) b_n) + P(mS_t^2 > t(m - 1) b_n) \]

\[ + \sum_{j=1}^{m-2} P(m(S_n - S_k)^2 > s(m - j) b_n, t(j - 1) b_n < mS_t^2 < jtb_n) \]
\[ P(mS_{n-k}^2 > s(m-1)b_n) + P(mS_i^2 > t(m-1)b_n) + m\phi(k-i) \]
\[ + \sum_{j=2}^{m-2} P(mS_i^2 > t(j-1)b_n) P(mS_{n-k}^2 > s(m-j)b_n) \]
\[ \leq n^{-r}(\log n)^{-1-r}. \]  
(3.8)

Since \( k - i \leq nt \varepsilon^2 \), another application of Lemma 3.3 gives
\[ P(||S_k^2 - S_i^2|| > 4t\sigma^2n \log n) \]
\[ \leq P(||S_{k-i}^2|| > 4t\varepsilon \sigma^2 n \log n) + P(||S_{k-i}|| > 2\sigma(r(k-i) \log n)^{1/2}) \]
\[ + P(||S_i|| > 2\sigma(r \log n)^{1/2}) \leq n^{-r}(\log n)^{-1-r}. \]  
(3.9)

The lemma now follows from (3.8) and (3.9).

**Proof of Theorem 1.** Without loss of generality we assume that \( \sigma = 1 \). By Lemmas 3.1, 3.2 and 3.3, the sequence \( \{f_i\} \) defined in (3.1) satisfies (2.1). Let \( T = \{t_i: 0 \leq i \leq m\}, 0 = t_0 < \cdots < t_m = 1 \). By (3.4) and Lemma 3.4 it follows that \( \{f_n^T\} \) is \( r \)-quickly contained in \( K_T^r \). By Proposition 2.1, it is enough to show that \( K_T^r \subset \partial_r(f^T) \).

Let \( g \in K_T^r, \eta = \min_{1 \leq i \leq m} (t_i - t_{i-1}) \) and \( 0 < \varepsilon < \eta n/8m(1+r) \). Then there exists \( h \in K_T^r \) such that \( ||h - g|| < \varepsilon \) and \( \sum_{i=1}^m l_i^2(t_i - t_{i-1})^{-1} < -4m(1+r) \varepsilon/\eta \), where \( l_i = h(t_i) - h(t_{i-1}) \). For \( 0 < \beta < \eta \varepsilon^2/2r \), we have by Lemmas 3.2 and 3.3 that

\[ P(||f_n^T - g||_\infty < 8m\varepsilon) \geq P(||f_n^T - h||_\infty < 4m\varepsilon) \]
\[ \geq P(\max_{j < m} |S_{[nt_j]} - S_{[nt_{j-1}] + [n\beta]} - l_j s_n| < 3\varepsilon s_n) \]
\[ - m\phi(\max_{j < n\beta} |S_j| > 3\varepsilon s_n) \]
\[ \geq \prod_{j=1}^m P(|S_{[nt_j]} - S_{[nt_{j-1}] + [n\beta]} - l_j s_n| < 2\varepsilon s_n) \]
\[ - m\phi([n\beta]) - O(s_n^{-2r}) \]
\[ \geq \exp \left( - \left( \sum_{j=1}^m l_j^2(t_j - t_{j-1})^{-1} + m\varepsilon \right) \log n \right) \]
\[ - O(s_n^{-2r}) \gg n^{-r}, \]
and hence \( g \in \partial_r(f^T) \). This completes the proof of Theorem 1.

**Theorem 2.** Let \( \{X_n\} \) be a stationary sequence of strong mixing random variables with \( a(n) \leq e^{-\theta n} \) for some \( \theta > 0 \) and \( E(X_1) = 0 \). Let \( E|X_i|^q < \infty \) for some \( q > 2 + 2r \) and \( \sigma > 0 \). Then the conclusions of Theorem 1 hold.
**Theorem 3.** Let \( \{X_n\} \) be a stationary sequence of bounded strong mixing random variables with \( \alpha(n) \leq n^{-6r-4} \), \( E(X_1) = 0 \) and \( \sigma > 0 \). Then the conclusions of Theorem 1 hold.

Theorem 3 is required to prove Theorem 5 given in the next section. Proofs of Theorems 2 and 3 are similar to that of Theorem 1 and so are omitted. Instead of Lemma 3.3, one uses Theorem 1.1 of Babu and Singh [2] and Lemma 3.5, respectively, for Theorems 2 and 3. See also Remark 3.3.

**Lemma 3.5.** Let \( \{X_n\} \) be a stationary sequence of bounded strong mixing random variables with \( E(X_1) = 0 \) and \( \sigma = 1 \). If \( \alpha(n) \leq n^{-4-\theta} \), for some \( \theta > 0 \), then there exists a \( \mu \) depending on \( \theta \) such that for all real \( u \),

\[
|P(S_n > un^{1/2}) - \Phi(-u)| \leq (n^{-\theta} + e^{-u^2/2}) n^{-\mu}.
\]

This lemma can be proved using an embedding theorem of Strassen [17]. We omit the proof.

Since \( K_1 \) is compact, we immediately have the following corollary from Lemma 4 of Lai [12].

**Corollary 3.1.** Under the conditions of either Theorems 1, 2 or 3,

\[
r\text{-lim sup}(2nr \log n)^{-1/2} S_n = \sigma
\]

and each element in \([-\sigma, \sigma]\) is an \( r \)-quick limit point of the sequence \((2nr \log n)^{-1/2} S_n\).

**4. Empirical Processes**

Let \( \{X_n\} \) be a stationary sequence of random variables. Let

\[
F_n(t) = \frac{1}{n} \# \{1 \leq i \leq n : X_i \leq t \}
\]

and

\[
G_n(t) = (n/2r \log n)^{1/2} (F_n(t) - t). \tag{4.1}
\]

If \( \{X_n\} \) is a strong mixing sequence with \( \sum_{n=1}^{\infty} \alpha(n) < \infty \), then the function \( R: [0, 1] \times [0, 1] \rightarrow (\infty, \infty) \) defined by

\[
R(s, t) = \text{cov}(I(X_1 \leq s), I(X_1 \leq t)) + \sum_{n=2}^{\infty} \text{cov}(I(X_1 \leq t), I(X_n \leq s))
\]

\[
+ \sum_{n=2}^{\infty} \text{cov}(I(X_1 \leq s), I(X_n \leq t)) \tag{4.2}
\]
converges absolutely and uniformly in \([0,1] \times [0,1]\), where \(I(A)\) denotes the indicator function of \(A\). So in this case it follows that \(R\) is a well-defined continuous function and that it is non-negative definite. Throughout this section \(R\) is assumed to be positive definite. The reproducing kernel Hilbert space (RKHS) of \(R\) is denoted by \(H(R)\). Since \(R\) is continuous, the unit ball \(B_R = \{x \in H(R) : \|x\|_R \leq 1\}\) is compact in \(C[0,1]\) (see Oodaira [14]), where \(\|x\|_R\) denotes the norm of \(H(R)\). For more details on RKHS see Aronszajn [1]. It is shown in this section that \(\partial_r(G) = B_R\) for certain mixing sequences \(\{X_n\}\).

**Theorem 4.** Let \(\{X_n\}\) be a stationary sequence of \(\phi\)-mixing random variables with \(P(X_1 \leq t) = t\) for \(0 \leq t \leq 1\) and with \(\phi(n) \leq n^{-\theta}\) for some \(\theta > \max(2, r)\). Then \(\{G_n\}\) defined in (4.1) is \(r\)-quickly contained in \(B_R\) and \(\partial_r(G) = B_R\), where \(R\) is as defined in (4.2).

In proving the theorem we follow the approach of Philipp [15]. We require the following two propositions and lemma.

**Proposition 4.1.** Let \(\{v_n\}\) be a sequence of random vectors in \(\mathbb{R}^m\) such that for each \(z \in \mathbb{R}^m\) with \(|z| = 1\), \(r\)-lim sup \(z'v_n = 1\), where for any two vectors \(x\) and \(z\), \(x'z\) denotes the usual inner product and \(|x| = \sqrt{x'x}\). Then \(\{v_n\}\) is \(r\)-quickly contained in \(\{z \in \mathbb{R}^m : |z| \leq 1\}\) and \(\{z \in \mathbb{R}^m : |z| = 1\} \subset \partial_r(v)\).

**Proof.** Clearly \(r\)-lim sup \(|v_n| \leq m\). Since \(\{z \in \mathbb{R}^m : |z| = 1\}\) is compact, for any \(\varepsilon > 0\) there exist \(z_1, \ldots, z_k\) in \(\mathbb{R}^m\) such that \(|z_i| = 1\) and \(\sup_{|z| = 1} \min_i |z - z_i| < \varepsilon\). So

\[
|v_n| = \sup_{|z| = 1} |z'v_n| \leq \min_i |z - z_i| \cdot |v_n| + \max_i |z_i'v_n|
\]

\[
\leq \varepsilon (|v_n|/2m) + \max_i |z_i'v_n|.
\]

This immediately gives that \(v_n\) is \(r\)-quickly contained in \(\{z \in \mathbb{R}^m : |z| \leq 1\}\). To complete the proof, let \(|z| = 1\). Since \(|z - v_n|^2 = 1 + |v_n|^2 - 2z'v_n\), it follows that for each \(\varepsilon > 0\),

\[
(z'v_n > 1 - \varepsilon) \subset (|v_n|^2 > 1 + \varepsilon) \cup (|z - v_n|^2 < 3\varepsilon).
\]

Thus

\[
W = \sup(n \geq 1 : z'v_n > 1 - \varepsilon)
\]

\[
\leq \sup(n \geq 1 : |z - v_n|^2 < 3\varepsilon) + \sup(n \geq 1 : |v_n|^2 > 1 + \varepsilon)
\]

\[
= Z + Y \quad \text{(say)}.
\]
Since $EY^r < \infty$ and $EW^r = \infty$, it follows that $EZ' = \infty$. So $z \in \partial_r(u)$. This completes the proof.

**Proposition 4.2.** Let $\{v_n\}$ be a sequence of random vectors in $\mathbb{R}^{m+1}$. Let $S$ be a $(m+1) \times (m+1)$ symmetric positive definite matrix. Let $Q$ be the $m \times m$ matrix obtained from $S$ by removing the last column and the last row and let $u_n = \pi v_n$, where $\pi$ denotes the projection to the first $m$ coordinates. If for each $y \in \mathbb{R}^{m+1}$, $r$-lim sup $y'v_n = (y'Sy)^{1/2}$, then $\partial_r(u) = \{z \in \mathbb{R}^m : z'Q^{-1}z < 1\}$.

**Proof.** Since $Q^{-1}$ is positive definite, we can write $Q^{-1} = J'J$, where $J$ is some nonsingular matrix. Let $z \in \mathbb{R}^m$ with $\|z\| = 1$. If $y = (Jz, 0)$, then we have $y'v_n = z'J'u_n$ and $r$-lim sup $z'J'u_n = (z'J'QJz)^{1/2} = \|z\| = 1$. By Proposition 4.1, we have that $\{z : \|z\| = 1\} \subset \partial_r(J'u) \subset \{z : \|z\| \leq 1\}$. Since $J$ is nonsingular, by Lemma 4 of Lai [12] it follows that $J'\partial_r(u) = \partial_r(J'u)$. So $\partial_r(u) \subset \{z : z'Q^{-1}z \leq 1\}$. By a similar argument it follows that $\partial_r(v) \supset \{x : x'S^{-1}x = 1\}$. Since $\pi$ is continuous $\pi\partial_r(v) \subset \partial_r(u)$, and hence we have

$$\{z : z'Q^{-1}z \leq 1\} \supset \partial_r(u) \supset \pi\{x : x'S^{-1}x = 1\}. \quad (4.3)$$

The proposition follows if we show

$$\pi\{x : x'S^{-1}x = 1\} \supset \{z : z'Q^{-1}z \leq 1\}. \quad (4.4)$$

To prove (4.4) let $S = (\begin{smallmatrix} Q & a \\ a' & c \end{smallmatrix})$. Note that

$$S^{-1} = \frac{1}{c - a'Q^{-1}a} \begin{pmatrix} Q^{-1}((c - a'Q^{-1}a)I + aa'Q^{-1}) & -Q^{-1}a \\ -a'Q^{-1} & 1 \end{pmatrix}$$

is positive definite and hence $c - a'Q^{-1}a > 0$. For any $z \in \mathbb{R}^m$, $h(y) = (y)'S^{-1}(y) = z'Q^{-1}z + (y - z'Q^{-1}z)^2/(c - a'Q^{-1}a)$ is a continuous function of $y$ on the interval $[z'Q^{-1}z, \infty)$ and $h(z'Q^{-1}z) = z'Q^{-1}z$. Since $h(y) \to \infty$ as $y \to \infty$, (4.4) follows. This completes the proof of Proposition 4.2.

**Lemma 4.1.** Under the conditions of Theorem 4, there exists a $d > 0$ such that, whenever $0 \leq t, s \leq 1$, $0 < b < 1$, $|t - s| \leq b$, $1 \leq m \leq n$, $H \geq 0$ and $0 < D \leq bn^{19/24}$,

$$P\left( \left\| \sum_{i=1}^m x_i + H(t, s) \right\| > DD \right) \ll n^{-r-4} + \exp(-8D^2n^{-1}b^{-1}), \quad (4.5)$$

where

$$x_j(t, s) = I(\min(t, s) \leq x_j \leq \max(t, s)) - |t - s|.$$

The lemma follows from the proof of Lemma 2.1 of Babu and Singh [4].
Proof of Theorem 4. By Lemma 4.1, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $E(\sup(n \geq 1: \omega(\delta, G_n) > \varepsilon))' < \infty$. Observe, as in the proof of Proposition 5.2.1 of Philipp [15], that for any finite subset $T$ of $[0, 1]$, $B_T^n = \{z: z'R_T^{-1}z \leq 1\}$, where $R_T$ is the matrix $((R(t, s)))$, $t, s \in T$. Now Corollary 3.1 and Propositions 2.1, 4.1 and 4.2 give the result.

Theorem 5. Let $\{X_n\}$ be a sequence of stationary strong mixing random variables with $P(X_1 < t) = t$ for $0 \leq t \leq 1$ and for some $\theta > \max(2r + 6, 6r + 4)$, $a(n) \propto n^{-\theta}$. Then the conclusions of Theorem 4 hold.

The proof of this theorem is similar to that of Theorem 4; instead of Lemma 4.1 we use Lemma 4.2 given below.

Lemma 4.2. Under the conditions of Theorem 5, there exist $d, \delta, \varepsilon > 0$ such that for $0 < t, s < 1$, $n^{-1} < b < 1$, $|t - s| < b$, and $H \geq 0$, we have

$$P\left(\left|\sum_{i=1}^n x_i + nH(t, s)\right| > db^4(n \log n)^{1/2}\right) \leq n^{-r^2 - \varepsilon},$$

where $x_j$ is as defined in (4.5).

We omit the proof of Lemma 4.2. It can be proved using an embedding theorem of Strassen [17].

If $\{X_n\}$ is a sequence of i.i.d. random variables with $P(X_1 < t) = t$ for $0 \leq t \leq 1$, then it follows that $\{G_n\}$ defined in (4.1) is $r$-quickly relatively compact in $D[0, 1]$ and that $\partial_r(G) = \{x \in K_1: x(1) = 0\}$. Further, by using Lemma 4 of Lai [12], we obtain that $r$-$\lim$ sup $\|G_n\| \leq 2^{-1}$.

Since $B_R$ is compact under the assumptions of Theorem 4 as well as those of Theorem 5, by Lemma 4 of Lai [12], we have the following corollaries.

Corollary 4.1. Under the assumptions of either Theorem 4 or 5, we have $r$-$\lim$ sup $\|G_n\| < c$, for some $c > 0$.

Corollary 4.2. Under the assumptions of either Theorem 4 or 5, if $w$ is a bounded function on $[0, 1]$, then $\{J_n\}$ is $r$-quickly contained in $\{w \cdot f: f \in B_R\}$ and this set equals $\partial_r(J)$, where $R$ is as defined in (4.2) and $J_n(t) = w(t)G_n(t)$.

The following two theorems give $r$-quick versions of Bahadur–Kiefer-type representations for quantiles.

For $v > 0$, let $F_n^{-1}(v) = \inf(y: F_n(y) \geq v)$ and $F_n^{-1}(0) = F_n^{-1}(0^+)$.
THEOREM 6. Let \( \{X_n\} \) be as in Theorem 4. Then there exists a constant \( b > 0 \) such that

\[
E(\sup_{n \geq 1} (\sup_{0 < t < 1} |F_n^{-1}(t) + F_n(t) - 2t| > b(n/\log n)^{-3/4}))^r < \infty.
\]

Proof. Let \( h_n = \sup_{0 < t < 1} |F_n(t) - t| \). Clearly \( h_n > 0 \) and for all \( t \in [0, 1] \), \( t - 2h_n < F_n(t) < t + 2h_n \). This implies, for all \( t \in [0, 1] \), that \( F_n(t - 2h) < t < F_n(t + 2h_n) \) and hence for all \( t \in [0, 1] \), we have \( t - 2h_n \leq F_n^{-1}(t) \leq t + 2h_n \). So by Corollary 4.1 it follows that for some \( c > 0 \),

\[
r\text{-lim sup } w_n < c,
\]

where \( w_n = \sup_{0 < t < 1} (n/\log n)^{1/2} |F_n^{-1}(t) - t| \).

On the set \( \{w_n < c\} \), we have as in Babu and Singh [4] that

\[
|F_n^{-1}(t) + F_n(t) - 2t| \\
\leq |F_n F_n^{-1}(t) - t| + |F_n F_n^{-1}(t) - F_n^{-1}(t) - F_n(t) + t| \\
\leq 2 |F_n F_n^{-1}(t) - F_n^{-1}(t) - F_n(t) + t| \\
+ |F_n(F_n^{-1}(t) - 0) - F_n^{-1}(t) - F_n(t) + t| \\
\leq 3 \sup_{0 < t < 1} \sup_{s < c(n/\log n)^{-1/2}} |F_n(t + s) - F_n(t) - s| \\
\leq 3 \max_{|i| \leq i_n} \max_{0 \leq i + j \leq j_n} |F_n((i + j) a_n) - F_n(j a_n) - i a_n| + 6a_n,
\]

where \( i_n = \lceil c(n/\log n)^{1/4} \rceil + 1 \), \( j_n = \lceil (n/\log n)^{3/4} \rceil + 1 \) and \( a_n = (n/\log n)^{-3/4} \). The result now follows from (4.6) and Lemma 4.1.

A similar proof using Lemma 4.2 instead of Lemma 4.1 gives the following theorem.

THEOREM 7. Let \( \{X_n\} \) be as in Theorem 5. Then there exists a \( \delta > 0 \) such that

\[
E(\sup_{n \geq 1} (\sup_{0 < t < 1} |F_n^{-1}(t) + F_n(t) - 2t| > n^{-\delta - 1/2}))^r < \infty.
\]

Even though Theorem 7 is much weaker than Theorem 6 (for \( \phi \)-mixing case) it is sufficient for concluding \( r \)-quick results for quantile processes from the same for empirical processes.
5. SOME APPLICATIONS

In this section, the results of previous section are extended to general distributions and asymptotic moments of first exit times of some linear functions of order statistics are computed.

Let $F$ be a distribution function such that for some interval $I$, $\inf\{F'(y): y \in I\} > 0$, $F'(y) = 0$ for $y \notin I$ and $\sup\{F''(y): y \in I\} < \infty$. Let $\{Y_n\}$ be a sequence of $\phi$-mixing random variables with $P(Y_1 \leq y) = F(y)$ and $\phi(n) \leq n^{-\theta}$ for some $\theta > \max(2, r)$. Following the lines of proof of Theorem 7 of Babu and Singh [4] and using Theorem 6, we have

**Theorem 8.** If $w$ is a bounded real-valued function on $[0, 1]$, then for some $d > 0$,

$$
\{w(F^{-1}(y)) - y\} \frac{n}{(n \log n)^{3/4}} < d.
$$

**Corollary 5.1.** Let $w$ and $\{Y_n\}$ be as in Theorem 8. Let $X_n = F(Y_n)$ and $g(y) = w(y) F'(F^{-1}(y))$. If $R$ defined in (4.2) is positive definite, then the sequence of processes $\{Z_n\}$, defined by $Z_n(t) = g(t)(F^{-1}(t) - F^{-1}(s))$ $(n/2r \log n)^{1/2}$, is $r$-quickly contained in $U = \{f \cdot g: f \in B_R\}$ and $\partial_r(Z) = U$.

Similarly one can show that, if $0 < t \leq s < 1$ are such that for some $\varepsilon > 0$, $F$ is twice differentiable on $[F^{-1}(s) - \varepsilon, F^{-1}(s) + \varepsilon]$ with the first derivative bounded away from zero and the second bounded on the interval, then

$$
L_n = \int_t^s (F^{-1}(u) - F^{-1}(u)) dw(u) = \frac{1}{n} \sum_{j=1}^n V_j + R_n,
$$

were $\omega$ is some function of bounded variation on $[t, s]$,

$$
V_j = \int_t^s (I(F(Y_j) \leq y) - y)(F'(F^{-1}(y)))^{-1} dw(y),
$$

and for some $d_1 > 0$

$$
\{R_n\} \frac{n}{(n \log n)^{3/4}} \leq d_1.
$$

The first exit time $N$ of $\{L_n\}$ is defined as

$$
N = \inf(n \geq 1: nL_n \notin (-a, b)).
$$

Let $\{Y_n\}$ be an i.i.d. sequence of random variables and let $L_n$ be defined as in (5.1). By Theorem 1 of Lai [11] and (5.2), it follows that as $a \to \infty$ and
$b \to \infty$ such that $a/(a + b) \to v, \sigma^2(a + b)^{-2} N$ converges in distribution to $\tau = \inf(y \geq 0: W(y) \leq (-v, 1 - v))$, where $\sigma^2$ is the variance of $V_1$ and $W$ is the standard Brownian motion. We further have

$$E(N^r) \sim ((a + b)/\sigma)^{2r} E(\tau^r). \quad (5.3)$$

If the third derivative of $w$ exists and is continuous, then this result can be concluded from the results of Lai [10].

Theorem 1 of Lai [11] can be extended to the strong mixing case when the random variables $\{X_n\}$ are bounded. Hence (5.3) holds even when the original sequence $\{Y_n\}$, with which we started, is strong mixing with $\alpha(n) \ll n^{-6r^{-4}}$.

**Remark 5.1.** Let $0 < t_0 < 1$. For some $c > 0$, $\eta > 1$, let the distribution function $F$ satisfy the condition

$$F(F^{-1}(t,v)) + v = F(F^{-1}(t,v)) = cv + O(v) \quad (5.4)$$

as $v \to 0$. If $\{Y_n\}$ is a strong mixing sequence with $\alpha(n) \ll n^{-6r^{-4}}$ and $P(Y_n \leq y) = F(y)$, then the asymptotic distribution of the first exit time of $F_n^{-1}(t_0)$ can be computed similarly. Many distribution functions $F$ of practical interest satisfy the condition (5.4). The following example is one such.

Consider the exponential distribution with density $\frac{1}{z}e^{-\frac{y}{z}}$, $y, z \in \mathbb{R}$. The maximum likelihood estimate of $z$ is the sample median. The density function is not differentiable at the population median $z$. However, by Remark 5.1, one can study the first exit time of the maximum likelihood estimator as above.

6. GAUSSIAN PROCESSES

In this section we characterize the $r$-quick limit sets of certain Gaussian processes in terms of the unit ball of the associated RKHS. We start with the following theorem.

**Theorem 9.** Let $\{Z_n\}$ be a sequence of Gaussian random variables with mean zero and variance one. If for each $\epsilon > 0$, there exist $p > 1, \eta > 1$ such that

$$P\left(\max_{n \leq m \leq n} |Z_n - Z_m| > \epsilon(\log n)^{1/2}\right) \ll n^{-r}(\log n)^{-\eta}, \quad (6.1)$$

then

$$r-\lim \sup(2r \log n)^{-1/2} Z_n = 1.$$
Proof. For $c^2 < r$, we have
\[ E(\sup(n \geq 1: Z_n > c(2 \log n)^{1/2}))^r \]
\[ = \sum_{n=1}^{\infty} n^{r-1} P(Z_n > c(2 \log n)^{1/2}) = \infty. \]

So
\[ r\text{-}\lim \sup(2r \log n)^{-1/2} Z_n \geq 1. \]  \hspace{1cm} (6.2)

To complete the proof, let $\varepsilon > 0$. For any $p > 1$ we have
\[ E(\sup(n \geq 1: Z_n > (2(r + \varepsilon) \log n)^{1/2}))^r \]
\[ \leq \sum_{n=1}^{\infty} P\left(\max_{\mu^{-1} \leq m < \mu^n} (2(r + \varepsilon) \log m)^{-1/2} Z_m > 1\right). \]  \hspace{1cm} (6.3)

Since, for $n \leq m \leq \rho n$,
\[ |(\log m)^{-1/2} Z_m - (\log n)^{-1/2} Z_n| \]
\[ \leq (\log m)^{-1/2} \left[ |Z_n - Z_m| + |Z_n| \left(\frac{\log m}{\log n} - 1\right)\right] \]
\[ \leq (\log n)^{-1/2} |Z_n - Z_m| + |Z_n| (\log n)^{-1}. \]  \hspace{1cm} (6.4)

The result follows from (6.1)-(6.4).

**Theorem 10.** Suppose $X(t), t \in [0, 1]$, is a separable real-valued Gaussian process with mean zero and positive definite continuous covariance kernel $\Gamma$ satisfying
\[ m(s) - X(t))^2 \leq \varphi(t - s), \quad 0 \leq t, s \leq 1, \]  \hspace{1cm} (6.5)

where $\varphi$ is a continuous nondecreasing function on $[0, 1]$ such that \[ \int_{-\infty}^{\infty} \varphi(e^{-s}) \, ds < \infty. \] Let $\{X_n\}$ be a sequence of Gaussian processes defined on the same probability space and having the same distribution as the process $X$, and let for each $\varepsilon > 0, 0 \leq t \leq 1$, there exist $\rho > 1, \eta > 1$ such that
\[ P(\max_{n \leq m \leq \rho n} |X_n(t) - X(t)| > \varepsilon(\log n)^{1/2}) \leq n^{-r}(\log n)^{-n}. \]  \hspace{1cm} (6.6)

If $Y_n = (2r \log n)^{-1/2} X_n$, then the sequence $\{Y_n\}$ is $r$-quickly contained in $B_\Gamma$, and $\partial_r(Y) = B_\Gamma$ where $B_\Gamma$ is the unit ball in the RKHS $H(\Gamma)$ of $\Gamma$. 


Proof of Theorem 10. By the corollary of Marcus [14, p. 307], for each 
\( \varepsilon > 0, 0 \leq t \leq 1 \), we have a \( \delta > 0 \) such that
\[
P\left( \sup_{t \leq s \leq \min(1, t + \delta)} |Y_n(t) - Y_n(s)| > \varepsilon \right) \leq n^{-r-2}.
\]
Since
\[
\omega(\delta, Y_n) \leq \max_{\delta i < 1} \sup_{i \delta < s < (i + 1) \delta} |Y_n(s) - Y_n(i \delta)|,
\]
it follows that
\[
E(\sup(n \geq 1: \omega(\delta, Y_n) > \varepsilon) \leq \sum_{n=1}^{\infty} n' P(\omega(\delta, Y_n) > \varepsilon) < \infty.
\] (6.7)
Further by Propositions 4.1 and 4.2 and Theorem 9, it follows that for each finite \( T \subset [0, 1) \), the sequence \{Y_n^T\} is \( r \)-quickly contained in \( B_T^1 \) and that \( \partial_r(Y^T) = B_T^r \). The theorem now follows from (6.7) and Proposition 2.1.

Corollary 6.1. Let \( W \) denote the standard Brownian motion on \( C[0, \infty) \). Let \( Y_n(t) = (2n \log n)^{-1/2} W(nt), n \geq 1, 0 \leq t \leq 1 \). Then for each \( r > 0 \), \{Y_n\} is \( r \)-quickly relatively compact in \( C[0, 1] \) and \( \partial_r(Y) = K_r \), where \( K_r \) is defined in Section 3.

Proof. Since for any \( \rho > 1 \), \( n \leq m \leq \rho n \),
\[
|m^{-1/2} W(mt) - n^{-1/2} W(nt)|
\leq m^{-1/2} |W(mt) - W(nt)| + (n^{-1/2} - m^{-1/2}) |W(nt)|
\leq n^{-1/2} (|W(mt) - W(nt)| + (\rho - 1) |W(nt)|),
\]
condition (6.6) is satisfied for \( X_n(t) = n^{-1/2} W(nt) \) for some \( \rho > 1 \). So the corollary follows from Theorem 10.

Remark 6.1. Similar result holds for the Brownian bridge.

Theorem 10 can be easily extended (as in Lai [9]) to Gaussian processes with multidimensional parameter. This generalization has an interesting application to the Kiefer process.

A Kiefer process \{\xi(y, t): y \geq 0, 0 \leq t \leq 1\} is a separable Gaussian process with \( E\xi(y, t) = 0 \), and covariance function \( \Gamma((y_1, t_1), (y_2, t_2)) = \min(y_1, y_2)(\min(t_1, t_2) - t_1 t_2) \). This process appears in the study of quantile processes. The process \( \xi \) can be represented in terms of standard Wiener processes. For more details see the recent work of Csörghó and Révész [7].

Let \{X_n\} be a sequence of processes defined on \( I_2 \) by,
\[
X_n(t, s) = n^{-1/2} \xi(nt, s) \quad \text{(6.8)}
\]
for \((t, s) \in I_2 = [0, 1] \times [0, 1]\). The restriction \(R\) of \(f\) to \(I_2\) is clearly the covariance function of \(X_1\). The unit ball \(K\) of the RKHS, \(H(R)\), is given by

\[
K = \left\{ h \in C(I_2): \text{for } (u, v) \in [0, 1] \times [0, 1], \int_0^s \int_0^t f(u, v) \, du \, dv; s, t \in [0, 1], \int_0^1 \int_0^1 f^2(u, v) \, du \, dv \leq 1 \right\}
\]

(6.9)

See Lai [9]. Thus we have the following corollary.

**Corollary 6.2.** Let \(\{X_n\}\) be the sequence defined in (6.8). Let \(Y_n = (2r \log n)^{-1/2} X_n\). The sequence \(\{Y_n\}\) is \(r\)-quickly relatively compact in \(C(I_2)\) and \(\partial_r(Y) = K\).

Corollary 6.2 has an interesting consequence. Let \(\{Z_n\}\) be a sequence of i.i.d. random variables with \(P(Z_1 \leq t) = t\) for \(0 < t < 1\). Let

\[
H_n(s, t) = \{1 \leq m \leq ns: Z_m \leq t\} - t[ns].
\]

On a suitable probability space there exist a Kiefer process \(\xi\) and a version of the sequence \(\{H_n\}\) such that for all \(z\) and \(n\),

\[
P(\sup_{0 \leq s \leq 1} \sup_{0 \leq t \leq 1} |H_n(s, t) - \xi([ns], t)| > (\eta \log n + z) \log n) < ye^{-\lambda z},
\]

where \(\eta, \gamma\) and \(\lambda\) are positive absolute constants (see Theorem A in the paper of Csörgő and Révész [7].) So it follows from Corollary 6.2 that the sequence \(\{(2r n \log n)^{-1/2} H_n\}\) is \(r\)-quickly relatively compact in \(C(I_2)\) and that the set of its \(r\)-quick limit points is the set \(K\) defined in (6.9).

**References**


