DEPARTMENT OF STATISTICS, Penn State University
Statistics 514 (Spring 2010)

Practice Problems

1. Suppose $X_1, \ldots, X_n$ are iid from a Poisson distribution with parameter θ.

   a. Find the limiting distribution of $\sqrt{n}(\bar{X} - \theta)$.

   b. State the limiting distribution of $\sqrt{n}(g(\bar{X}) - g(\theta))$, assuming that $g(u)$ is sufficiently smooth.

   c. Find $g(u)$ so that the asymptotic variance in part b does not depend on θ. (Then $g(u)$ is called a variance stabilizing transformation.)

   d. Use the result in part c to construct an approximate 95% CI for θ.

2. Assume the number of plants of a certain species in a region R has a Poisson distribution with parameter $\lambda a(R)$ where $a(R)$ is the area of region R. To make inferences about the unknown parameter λ, an ecologist selects n non-overlapping regions $R_1, \ldots, R_n$, all of the same area $a(R)$, and counts the number of plants found in each region. Let $X_1, \ldots, X_n$ denote these counts.

   a. Give the definition of a consistent estimate $\hat{\theta}$ of the parameter θ.

   b. Let $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$. Show that $\bar{X}$ is a consistent estimator of $\lambda a(R)$.

   c. Show that if T is also a consistent estimator of $\lambda a(R)$, then so is $n^{-1} \sum_{i=1}^{n} (X_i - T)^2$.

   d. Let $U_i = I(X_i = 0)$. Thus $U_i = 1$ if $X_i = 0$ and $U_i = 0$ if $X_i > 0$. Show that $T = -log(n^{-1} \sum_{i=1}^{n} U_i)$ is a consistent estimator of $\lambda a(R)$.

   e. Derive the asymptotic distributions of $\sqrt{n}(\bar{X} - \lambda a(R))$ and $\sqrt{n}(T - \lambda a(R))$.

3. Let Y_1, Y_2, Y_3 be the order statistics of a random sample size 3 from a distribution having pdf $f(x) = 1$ for $0 < x < 1$, zero elsewhere. Find the distribution of the sample range $Z = Y_3 - Y_1$.

4. Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x; \theta) = e^{-x/\theta}/\theta$ for $0 < x < \infty$, zero elsewhere. Find the mle of $P(X \leq 2)$.

5. Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x; \theta) = 2x/\theta^2$, $0 < x \leq \theta$, zero elsewhere.

   a. Find the mle $\hat{\theta}$ for θ.

   b. Prove or disprove that $\hat{\theta}_{\text{mle}}$ is unbiased.

c. Find the mle for the median of the distribution.

6. Let $X_1, \ldots, X_n$ be a random sample from a gamma distribution with $\alpha = 4$ and $\beta = \theta > 0$. (Note that the pdf $f(x) = x^{\alpha-1}e^{-x/\beta}/(\Gamma(\alpha)\beta^\alpha)$ for $0 < x < \infty$, $EX = \alpha\beta$ and $VarX = \alpha\beta^2$.)

   a. Find the mle of θ.

   b. Prove or disprove that $\hat{\theta}_{\text{mle}}$is efficient.

   c. Find the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$.

   d. Find the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)/\hat{\theta}$.

7. Suppose $X_1, \ldots, X_n$ are iid with $f(x) = \theta x^{\theta-1}$ for $0 < x < 1$, $\theta > 0$.

   a. What is EX? What is $VarX$?

   b. Suppose we estimate θ by $\hat{\theta} = \bar{X}/(1 - \bar{X})$. What is the large sample (i.e. asymptotic) distribution of $\hat{\theta}$.

8. Suppose you have a random number generator which can generate normal and χ^2 random variables. Describe how you would generate (i) a sample of iid random variables having a t_ν distribution, and (ii) a sample of iid random variables having a F_{ν_1,ν_2} distribution.

   a. The answer to the above random number generation question is based on a definition of the t and F distributions. Give these definitions. Now let $X_1, \ldots, X_m$ be iid from a normal distribution with mean μ and variance σ^2, and $Y_1, \ldots, Y_n$ be iid from a normal distribution with mean μ and variance σ^2. Let $\bar{X}$, $\bar{Y}$, s_1^2, s_2^2 be the sample means and variances from the two samples, and let $s_p^2 = \{(m-1)s_1^2 + (n-1)s_2^2\}/(m+n-2)$ be the pooled estimator of the common variance σ^2.

   b. Show that

$$T = \frac{\bar{X} - \bar{Y}}{s_p\sqrt{\frac{1}{m} + \frac{1}{n}}}$$

   has a t_{m+n-2} distribution.

   c. Show that

$$F = \frac{s_1^2}{s_2^2}$$

   has an $F_{m-1,n-1}$ distribution.

d. Now suppose both m and n tend to infinity. Find the asymptotic distribution of T assuming only that the X's and Y's have the same variance, i.e. same setting as in Part (a) but without the normality assumption.

9. Let $X_1, \ldots, X_n$ be iid Bernoulli(p). The object is to estimate $\theta = 1/p$.

   a. Find the maximum likelihood estimator of θ.

   b. What is the asymptotic distribution of the MLE as $n \to \infty$?

10. Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ^2 given an iid random sample $X_1, \ldots, X_n$ from density $f_\theta(x) = \theta^{1/2} exp(-\theta^{1/2} x)$, where $0 \le x$, $0 \le \theta$.

11. Suppose that $X_1, X_2, \ldots, X_{2n}$ are independently distributed as $X_i \sim N(0, \sigma^2)$ for $i = 1, \ldots, n$ and $X_i \sim N(0, 2\sigma^2)$ for $i = n+1, \ldots, 2n$. Find $\hat{\sigma}^2$, the maximum likelihood estimate of σ^2. What is the distribution of $\hat{\sigma}^2$?

12. Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x, \theta) = \{\theta^3 x^2 exp(-\theta x)\}/2$ for $x > 0$, $\theta > 0$.

    a. Find the mle of θ.

    b. Prove that $\hat{\theta} \to \theta$ in probability. (It is consistent).

    c. Find the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$.

    d. Find the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)/\hat{\theta}$.

13. Consider a sequence of random variables X_n for which $E(X_n) \to \mu$ for a fixed value $\mu \in R$ and $Var(X_n) \to 0$ as $n \to \infty$. Show that this implies $X_n \to \mu$ in probability for $n \to \infty$.

14. Let $W_1 < W_2 < \ldots < W_n$ be the order statistics of n independent observations from a $U(0, 1)$ distribution.

    a. For $1 \le r \le n$, derive the pdf of W_r, the rth order statistics.

    b. Use the pdf you've found in a. to find $E(W_1)$ and $E(W_n)$.

    c. Let X_1, X_2, X_3 be three independent observations from $U(0, 1)$ distribution. What is $P(X_1 < X_2 < X_3)$?

15. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a distribution that is $\mathcal{N}(\mu, \sigma^2)$.

    a. Find the maximum-likelihood estimator for μ.

    b. Is $\hat{\mu}_{mle}$ efficient?

c. Find the asymptotic distribution of the maximum likelihood estimator of $\theta = \mu^2$ when $\mu \neq 0$. Prove or disprove that $\hat{\theta}_{mle}$ is unbiased?

d. Find the asymptotic distribution of

$$\frac{\sqrt{n}(\hat{\theta}_{mle} - \theta)}{\sum(X_i - \bar{X})^2/n}.$$

Justify all your steps from scratch, carefully name all results used.

e. An alternative estimator of θ is $\bar{X}^2 - \sum(X_i - \bar{X})^2/\{n(n-1)\}$. Prove or disprove that this estimator is unbiased?