7.22 a) $$f(\bar{x},\theta) = f(\bar{x}|\theta)\pi(\theta) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-n(\bar{x}-\theta)^2/(2\sigma^2)} \frac{1}{\sqrt{2\pi}\tau} e^{-(\theta-\mu)^2/2\tau^2}.$$

b) Factor the exponent in part a) as

$$\frac{-n}{2\sigma^2}(\bar{x}-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2 = -\frac{1}{2v^2}(\theta-\delta(x))^2 - \frac{1}{\tau^2+\sigma^2/n}(\bar{x}-\mu)^2,$$

where $\delta(x) = (\tau^2\bar{x} + (\sigma^2/n)\mu)/(\tau^2+\sigma^2/n)$ and $v = (\sigma^2\tau^2/n)\big/(\tau+\sigma^2/n)$. Let $n(a,b)$ denote the pdf of a normal distribution with mean a and variance b. The above factorization shows that

$$f(x,\theta) = n(\theta,\sigma^2/n) \times n(\mu,\tau^2) = n(\delta(x),v^2) \times n(\mu,\tau^2+\sigma^2/n),$$

where the marginal distribution of $\bar{X}$ is $n(\mu,\tau^2+\sigma^2/n)$ and the posterior distribution of $\theta|x$ is $n(\delta(x),v^2)$. This also completes part c).

7.23 Let $t = s^2$ and $\theta = \sigma^2$. Since $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$, we have

$$f(t|\theta) = \frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}}\left(\frac{n-1}{\theta}t\right)^{[(n-1)/2]-1} e^{-(n-1)t/2\theta}\frac{n-1}{\theta}.$$

With $\pi(\theta)$ as given, we have (ignoring terms that do not depend on θ)

$$\pi(\theta|t) \propto \left[\left(\frac{1}{\theta}\right)^{((n-1)/2)-1} e^{-(n-1)t/2\theta}\frac{1}{\theta}\right]\left[\frac{1}{\theta^{\alpha+1}}e^{-1/\beta\theta}\right]$$

$$\propto \left(\frac{1}{\theta}\right)^{((n-1)/2)+\alpha+1} \exp\left\{-\frac{1}{\theta}\left[\frac{(n-1)t}{2}+\frac{1}{\beta}\right]\right\},$$

which we recognize as the kernel of an inverted gamma pdf, IG(a,b), with $a = ((n-1)/2)+\alpha$ and

$b = \left[\frac{(n-1)t}{2}+\frac{1}{\beta}\right]^{-1}$. Direct calculation shows that the mean of an IG(a,b) is $1/((a-1)b)$, so

$$E(\theta|t) = \frac{\frac{n-1}{2}t+\frac{1}{\beta}}{\frac{n-1}{2}+\alpha-1} = \frac{\frac{n-1}{2}s^2+\frac{1}{\beta}}{\frac{n-1}{2}+\alpha-1}.$$

This is a Bayes estimator of σ^2.

7.24 For n observations, $Y = \Sigma X_i \sim \text{Poisson}(n\lambda)$.

a) The marginal pmf of Y is

$$m(y) = \int_0^\infty \frac{(n\lambda)^y e^{-n\lambda}}{y!}\frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}e^{-\lambda/\beta}\,d\lambda$$

$$= \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \lambda^{(y+\alpha)-1}e^{-\frac{\lambda}{\beta/(n\beta+1)}}\,d\lambda = \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha}\cdot\Gamma(y+\alpha)\cdot\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}.$$

Thus,

$$\pi(\lambda|y) = \frac{f(y|\lambda)\pi(\lambda)}{m(y)} = \frac{\lambda^{(y+\alpha)-1}e^{-\frac{\lambda}{\beta/(n\beta+1)}}}{\Gamma(y+\alpha)\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}} \sim \text{gamma}\left(y+\alpha,\frac{\beta}{n\beta+1}\right).$$

b) $$E(\lambda|y) = (y+\alpha)\frac{\beta}{n\beta+1} = \frac{\beta}{n\beta+1}y + \frac{1}{n\beta+1}(\alpha\beta).$$

$$\text{Var}(\lambda|y) = (y+\alpha)\frac{\beta^2}{(n\beta+1)^2}.$$

9.53 ~~1016~~ a) $E\left[b\,\mathrm{Len}(C) - I_C(\mu)\right] = 2c\sigma b - P(|Z| \le c)$, where $Z \sim n(0,1)$.

b) $\frac{d}{dc}\left[2c\sigma b - P(|Z| \le c)\right] = 2\sigma b - 2\left(\frac{1}{\sqrt{2\pi}} e^{-c^2/2}\right)$.

c) If $b\sigma > 1/\sqrt{2\pi}$ the derivative is always positive since $e^{-c^2/2} < 1$.

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$$\begin{aligned} E[L((\mu, \sigma), C)] &= E\left[L((\mu, \sigma), C) \mid S < K\right] P(S < K) + E\left[L((\mu, \sigma), C) \mid S > K\right] P(S > K) \\ &= E\left[L((\mu, \sigma), C') \mid S < K\right] P(S < K) + E\left[L((\mu, \sigma), C) \mid S > K\right] P(S > K) \\ &= R\left[L((\mu, \sigma), C')\right] + E\left[L((\mu, \sigma), C) \mid S > K\right] P(S > K), \end{aligned}$$

where the last equality follows because C' is $\emptyset$ if $S > K$.

The conditional expectation in the second term is bounded by

$$\begin{aligned} E\left[L((\mu, \sigma), C) \mid S > K\right] &= E\left[b\,\mathrm{Len}(C) - I_C(\mu) \mid S > K\right] \\ &= E\left[2bcS - I_C(\mu) \mid S > K\right] \\ &> E\left[2bcK - 1 \mid S > K\right] \qquad \text{(since } S > K \text{ and } I_C \le 1\text{)} \\ &= 2bcK - 1, \end{aligned}$$

which is positive if $K > 1/2bc$. For those values of K, C' dominates C.