8.14 The CLT tells us that $Z = (\sum_i X_i - np)/\sqrt{np(1-p)}$ is approximately n(0, 1). For a test that rejects H_0 when $\sum_i X_i > c$, we need to find c and n to satisfy

$$P\left(Z > \frac{c-n(.49)}{\sqrt{n(.49)(.51)}}\right) = .01 \quad \text{and} \quad P\left(Z > \frac{c-n(.51)}{\sqrt{n(.51)(.49)}}\right) = .99.$$

We thus want

$$\frac{c-n(.49)}{\sqrt{n(.49)(.51)}} = 2.33 \quad \text{and} \quad \frac{c-n(.51)}{\sqrt{n(.51)(.49)}} = -2.33.$$

Solving these equations gives $n = 13{,}567$ and $c = 6{,}783.5$.

8.15 From the Neyman-Pearson lemma the UMP test rejects H_0 if

$$\frac{f(x \mid \sigma_1)}{f(x \mid \sigma_0)} = \frac{(2\pi\sigma_1^2)^{-n/2} e^{-\Sigma_i x_i^2/(2\sigma_1^2)}}{(2\pi\sigma_0^2)^{-n/2} e^{-\Sigma_i x_i^2/(2\sigma_0^2)}} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{\frac{1}{2}\sum_i x_i^2\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right\} > k$$

for some $k \geq 0$. After some algebra, this is equivalent to rejecting if

$$\sum_i x_i^2 > \frac{2\log\left(k\left(\sigma_1/\sigma_0\right)^n\right)}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)} = c \quad \left(\text{because } \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0\right).$$

This is the UMP test of size α, where $\alpha = P_{\sigma_0}(\sum_i X_i^2 > c)$. To determine c to obtain a specified α, use the fact that $\sum_i X_i^2/\sigma_0^2 \sim \chi_n^2$. Thus

$$\alpha = P_{\sigma_0}\left(\sum_i X_i^2/\sigma_0^2 > c/\sigma_0^2\right) = P\left(\chi_n^2 > c/\sigma_0^2\right),$$

so we must have $c/\sigma_0^2 = \chi^2_{n,\alpha}$, which means $c = \sigma_0^2\chi^2_{n,\alpha}$.

8.16 a.

$$\begin{aligned} \text{Size} &= P(\text{reject } H_0 \mid H_0 \text{ is true}) = 1 \Rightarrow \text{Type I error} = 1. \\ \text{Power} &= P(\text{reject } H_0 \mid H_A \text{ is true}) = 1 \Rightarrow \text{Type II error} = 0. \end{aligned}$$

b.

$$\begin{aligned} \text{Size} &= P(\text{reject } H_0 \mid H_0 \text{ is true}) = 0 \Rightarrow \text{Type I error} = 0. \\ \text{Power} &= P(\text{reject } H_0 \mid H_A \text{ is true}) = 0 \Rightarrow \text{Type II error} = 1. \end{aligned}$$

8.19 The pdf of Y is

$$f(y|\theta) = \frac{1}{\theta} y^{(1/\theta)-1} e^{-y^{1/\theta}}, \quad y > 0.$$

By the Neyman-Pearson Lemma, the UMP test will reject if

$$\frac{1}{2} y^{-1/2} e^{y - y^{1/2}} = \frac{f(y|2)}{f(y|1)} > k.$$

To see the form of this rejection region, we compute

$$\frac{d}{dy}\left(\frac{1}{2}y^{-1/2}e^{y-y^{1/2}}\right) = \frac{1}{2}y^{-3/2}e^{y-y^{1/2}}\left(y - \frac{y^{1/2}}{2} - \frac{1}{2}\right)$$

which is negative for $y < 1$ and positive for $y > 1$. Thus $f(y|2)/f(y|1)$ is decreasing for $y \leq 1$ and increasing for $y \geq 1$. Hence, rejecting for $f(y|2)/f(y|1) > k$ is equivalent to rejecting for $y \leq c_0$ or $y \geq c_1$. To obtain a size α test, the constants c_0 and c_1 must satisfy

$$\alpha = P(Y \leq c_0|\theta = 1) + P(Y \geq c_1|\theta = 1) = 1 - e^{-c_0} + e^{-c_1} \quad \text{and} \quad \frac{f(c_0|2)}{f(c_0|1)} = \frac{f(c_1|2)}{f(c_1|1)}.$$

Solving these two equations numerically, for $\alpha = .10$, yields $c_0 = .076546$ and $c_1 = 3.637798$. The Type II error probability is

$$P(c_0 < Y < c_1|\theta = 2) = \int_{c_0}^{c_1} \frac{1}{2}y^{-1/2}e^{-y^{1/2}}\,dy = -e^{-y^{1/2}}\Big|_{c_0}^{c_1} = .609824.$$

8.49 a. The p-value is

$$
\begin{aligned}
& P\left\{\left(\begin{array}{c}\text{7 or more successes}\\ \text{out of 10 Bernoulli trials}\end{array}\right)\Bigg|\,\theta=\frac{1}{2}\right\} \\
= & \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3+\binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2+\binom{10}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1+\binom{10}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0 \\
= & .171875.
\end{aligned}
$$

b.

$$
\begin{aligned}
\text{P-value} &= P\{X\ge 3 \mid \lambda=1\} = 1-P(X<3\mid\lambda=1)\\
&= 1-\left[\frac{e^{-1}1^2}{2!}+\frac{e^{-1}1^1}{1!}+\frac{e^{-1}1^0}{0!}\right] \approx .0803.
\end{aligned}
$$

c.

$$
\begin{aligned}
\text{P-value} &= P\{\sum_i X_i \ge 9 \mid 3\lambda=3\} = 1-P(Y<9\mid 3\lambda=3)\\
&= 1-e^{-3}\left[\frac{3^8}{8!}+\frac{3^7}{7!}+\frac{3^6}{6!}+\frac{3^5}{5!}+\cdots+\frac{3^1}{1!}+\frac{3^0}{0!}\right] \approx .0038,
\end{aligned}
$$

where $Y=\sum_{i=1}^3 X_i \sim \text{Poisson}(3\lambda)$.

10.32 a. First calculate the MLEs under $p_1=p_2=p$. We have

$$
L(p|x)=p^{x_1}p^{x_2}p^{x_3}\cdots p_{n-1}^{x_{n-1}}\left(1-2p-\sum_{i=3}^{n-1}p_i\right)^{m-x_1-x_2-\cdots-x_{n-1}}.
$$

Taking logs and differentiating yield the following equations for the MLEs:

$$
\begin{aligned}
\frac{\partial \log L}{\partial p} &= \frac{x_1+x_2}{p}-\frac{2\left(m-\sum_{i=1}^{n-1}x_i\right)}{1-2p-\sum_{i=3}^{n-1}p_i}=0\\
\frac{\partial \log L}{\partial p_i} &= \frac{x_i}{p_i}-\frac{x_n}{1-2p-\sum_{i=3}^{n-1}p_i}=0, \quad i=3,\ldots,n-1,
\end{aligned}
$$

with solutions $\hat p=\frac{x_1+x_2}{2m}$, $\hat p_i=\frac{x_i}{m}, i=3,\ldots,n-1$, and $\hat p_n=\left(m-\sum_{i=1}^{n-1}x_i\right)/m$. Except for the first and second cells, we have expected = observed, since both are equal to x_i. For the first two terms, expected $=m\hat p=(x_1+x_2)/2$ and we get

$$
\sum\frac{(observed-expected)^2}{expected}=\frac{\left(x_1-\frac{x_1+x_2}{2}\right)^2}{\frac{x_1+x_2}{2}}+\frac{\left(x_2-\frac{x_1+x_2}{2}\right)^2}{\frac{x_1+x_2}{2}}=\frac{(x_1-x_2)^2}{x_1+x_2}.
$$

b. Now the hypothesis is about conditional probabilities is given by H_0: P(change—initial agree)=P(change—initial disagree) or, in terms of the parameters $H_0: \frac{p_1}{p_1+p_3}=\frac{p_2}{p_2+p_4}$. This is the same as $p_1p_4=p_2p_3$, which is not the same as $p_1=p_2$.