Exercise 5.11.

Solution. Let $g(s) = s^2$. Since $g(\cdot)$ is a convex function, we know from Jensen's inequality that $Eg(S) \geq g(ES)$, which implies $\sigma^2 = ES^2 \geq (ES)^2$. Taking square roots, $\sigma \geq ES$. From the proof of Jensen's Inequality, it is clear that, in fact, the inequality will be strict unless there is an interval I such that g is linear on I and $P(X \in I) = 1$. Since s^2 is "linear" only on single points, we have $ET^2 > (ET)^2$ for any random variable T, unless $P(T = ET) = 1$.

Exercise 5.29.

Solution. Let X_i = weight of i th booklet in package. The X_i s are iid with $EX_i = 1$ and $VarX_i = .05^2$. We want to approximate $P(\sum\limits_{i=1}^{100} X_i > 100.4) = P(\sum\limits_{i=1}^{100} X_i/100 > 1.004) = P(\overline{X} > 1.004)$. By the CLT, $P(\overline{X} > 1.004) \approx P(Z > (1.004 - 1)/(.05/10)) = P(Z > .8) = .2119$

Exercise 5.32.

Solution. a. For any $\epsilon > 0$,

$$P(|\sqrt{X_n} - \sqrt{a}| > \epsilon) = P(|\sqrt{X_n} - \sqrt{a}||\sqrt{X_n} + \sqrt{a}| > \epsilon|\sqrt{X_n} + \sqrt{a}|) =$$

$$= P(|X_n - a| > \epsilon|\sqrt{X_n} + \sqrt{a}|) \leq P(|X_n - a| > \epsilon\sqrt{a}) \to 0$$

as $n \to \infty$, since $X_n \to a$ in probability. Thus $\sqrt{X_n} \to \sqrt{a}$ in probability.

b. For any $\epsilon > 0$,

$$P(|\frac{a}{X_n} - 1| \leq \epsilon) = P(\frac{a}{1+\epsilon} \leq X_n \leq \frac{a}{1-\epsilon}) =$$

$$= P(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1-\epsilon}) \geq$$

$$\geq P(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1+\epsilon}) = P(|X_n - a| \leq \frac{a\epsilon}{1+\epsilon}) \to 1$$

as $n \to \infty$, since $X_n \to a$ in probability. Thus $\frac{a}{X_n} \to 1$ in probability.

c. $S_n^2 \to \sigma^2$ in probability. By a), $S_n = \sqrt{S_n^2} \to \sqrt{\sigma^2} = \sigma$ in probability. By b), $\frac{\sigma}{S_n} \to 1$ in probability.

Exercise 5.34.

Solution. Using $E\overline{X}_n = \mu$ and $Var\overline{X}_n = \frac{\sigma^2}{n}$, we obtain

$$E\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}}{\sigma}E(\overline{X}_n - \mu) = \frac{\sqrt{n}}{\sigma}(\mu - \mu) = 0.$$

$$Var\frac{\sqrt{n}(\overline{X}_n-\mu)}{\sigma}=\frac{n}{\sigma^2}Var(\overline{X}_n-\mu)=\frac{n}{\sigma^2}Var(\overline{X}_n)=\frac{n}{\sigma^2}\frac{\sigma^2}{n}=1.$$