Assignment 10

Exercise 4.8  Prove that (4.13) implies both (4.12) and (4.14) (the “forward half” of the Lindeberg-Feller Theorem). Use the following steps:

(a)  Prove that for any complex numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ with $|a_i| \leq 1$ and $|b_i| \leq 1$,

$$|a_1 \cdots a_n - b_1 \cdots b_n| \leq \sum_{i=1}^{n} |a_i - b_i|. \quad (4.19)$$

Hint:  First prove the identity when $n = 2$, which is the key step. Then use mathematical induction.

Sketch of solution:  We proceed by mathematical induction, starting with the case $n = 2$ (notice that the statement is completely trivial for $n = 1$, and we shall see that the induction step requires $n = 2$ anyway).

For $n = 2$, we must prove that $|a_1 a_2 - b_1 b_2| \leq |a_1 - b_1| + |a_2 - b_2|$. This follows immediately from the triangle inequality, together with the fact that $|a_i| \leq 1$ and $|b_i| \leq 1$ for all $i$:

$$|a_1 a_2 - b_1 b_2| = |a_1 a_2 - b_1 a_2 + b_1 a_2 - b_1 b_2| \leq |a_2||a_1 - b_1| + |b_1||a_2 - b_2| \leq |a_1 - b_1| + |a_2 - b_2|.$$ 

For a general $n > 2$, we let $c = a_1 a_2 \cdots a_{n-1}$ and $d = b_1 b_2 \cdots b_{n-1}$.

Then the argument for $n = 2$ gives

$$|ca_n - db_n| \leq |c - d| + |a_n - b_n|.$$ 

By the inductive step, we conclude that

$$|c - d| \leq \sum_{i=1}^{n-1} |a_i - b_i|.$$ 

Combining the previous two inequalities gives the desired result:

$$|ca_n - db_n| \leq \sum_{i=1}^{n} |a_i - b_i|. \quad 207$$
(b) Prove that
\[
\left| \phi_{Y_{ni}} \left( \frac{t}{s_n} \right) - \left( 1 - \frac{t^2 \sigma_{ni}^2}{2s_n^2} \right) \right| \leq \frac{\epsilon}{s_n^2} \left| Y_{ni}^2 I\{|Y_{ni}| \geq \epsilon s_n \} \right|.
\] (4.20)

**Hint:** Use the results of Exercise 1.43, parts (c) and (d), to argue that for any \( Y \),
\[
\left| \exp \left\{ \frac{itY_{ni}}{s_n} \right\} - \left( 1 + \frac{itY_{ni} - t^2 Y_{ni}^2}{2s_n^2} \right) \right| \leq \left| \frac{tY_{ni}}{s_n} \right|^3 I\left\{|Y_{ni}| < \epsilon s_n \right\} + \left( \frac{tY_{ni}}{s_n} \right)^2 I\{|Y_{ni}| \geq \epsilon s_n \}.
\]

**Sketch of solution:** Suppose \( t \) is arbitrary and \( \epsilon > 0 \) is fixed. Because \( EY_{ni} = 0 \) and \( |EZ| \leq E|Z| \) for any complex-valued random variable \( Z \) (a fact that is essentially the triangle inequality), we obtain
\[
\left| \phi_{Y_{ni}} \left( \frac{t}{s_n} \right) - \left( 1 - \frac{t^2 \sigma_{ni}^2}{2s_n^2} \right) \right| = \left| E \exp \left\{ \frac{itY_{ni}}{s_n} \right\} - \left( 1 + \frac{itY_{ni} - t^2 Y_{ni}^2}{2s_n^2} \right) \right|
\leq E \left| \exp \left\{ \frac{itY_{ni}}{s_n} \right\} - \left( 1 + \frac{itY_{ni} - t^2 Y_{ni}^2}{2s_n^2} \right) \right|.
\]

Using parts (c) and (d) of Exercise 1.43, we may argue that (ignoring the factor of 1/6 in part (c) does not change the inequality) for any random variable \( X \),
\[
|\exp\{itX\} - (1 + itX - tX^2/2)| \leq \min\{|tX|^3, |tX|^2\}
= \min\{|tX|^3, |tX|^2\} (I\{|X| < \epsilon\} + I\{|X| \geq \epsilon\})
\leq |tX|^3 I\{|X| < \epsilon\} + (tX)^2 I\{|X| \geq \epsilon\}
\leq \epsilon |t|^3 X^2 + (tX)^2 I\{|X| \geq \epsilon\}.
\]

Now, we let \( X = Y_{ni}/s_n \) and combine the previous two inequalities to conclude that
\[
\left| \phi_{Y_{ni}} \left( \frac{t}{s_n} \right) - \left( 1 - \frac{t^2 \sigma_{ni}^2}{2s_n^2} \right) \right| \leq E \left( \frac{\epsilon |t|^3 Y_{ni}^2}{s_n^2} + \frac{t^2 Y_{ni}^2}{s_n^2} I\{|Y_{ni}| \geq \epsilon s_n \} \right)
\]
\[
= \frac{\epsilon |t|^3 \sigma_{ni}^2}{s_n^2} + \frac{t^2}{s_n^2} E \left( Y_{ni}^2 I\{|Y_{ni}| \geq \epsilon s_n \} \right).
\]

(c) Prove that (4.13) implies (4.14).

**Hint:** For any \( i \), show that
\[
\frac{\sigma_{ni}^2}{s_n^2} < \epsilon^2 + \frac{E(Y_{ni}^2 I\{|Y_{ni}| \geq \epsilon s_n \})}{s_n^2}.
\]

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Sketch of solution: For an arbitrary $\delta > 0$—I’ll use $\delta$ instead of $\epsilon$ because $\epsilon$ has already been fixed in part (b)—since

$$Y_{ni}^2 = Y_{ni}^2 I\{|Y_{ni}| < \delta s_n\} + Y_{ni}^2 I\{|Y_{ni}| \geq \delta s_n\} < \delta^2 s_n^2 + Y_{ni}^2 I\{|Y_{ni}| \geq \delta s_n\}$$

taking expectations and dividing by $s_n^2$ gives

$$\frac{\sigma_{ni}^2}{s_n^2} < \delta^2 + \frac{E(Y_{ni}^2 I\{|Y_{ni}| \geq \delta s_n\})}{s_n^2}.$$ 

We may take the maximum of each side for all $1 \leq i \leq n$:

$$\max_{1 \leq i \leq n} \frac{\sigma_{ni}^2}{s_n^2} < \delta^2 + \max_{1 \leq i \leq n} \frac{E(Y_{ni}^2 I\{|Y_{ni}| \geq \delta s_n\})}{s_n^2}.$$ 

But the second term on the right hand side above is certainly less than

$$\frac{1}{s_n^2} \sum_{i=1}^{n} E(Y_{ni}^2 I\{|Y_{ni}| \geq \delta s_n\})$$,

which tends to zero because the Lindeberg condition is satisfied. We conclude that

$$\limsup_n \max_{1 \leq i \leq n} \frac{\sigma_{ni}^2}{s_n^2} \leq \delta^2.$$ 

Since $\delta$ is arbitrary, (4.13) must hold.

(d) Use parts (a) and (b) to prove that, for $n$ large enough so that $t^2 \max_i \sigma_{ni}^2/s_n^2 \leq 1$,

$$\left| \prod_{i=1}^{n} \phi_{Y_{ni}} \left( \frac{t}{s_n} \right) - \prod_{i=1}^{n} \left( 1 - \frac{t^2 \sigma_{ni}^2}{2s_n^2} \right) \right| \leq \epsilon |t|^3 + \frac{t^2}{s_n^2} \sum_{i=1}^{n} E(Y_{ni}^2 I\{|Y_{ni}| \geq \epsilon s_n\}).$$

Sketch of solution: Since we know that (4.13) holds, we know that there exists $N$ such that for all $n > N$,

$$t^2 \max_{1 \leq i \leq n} \frac{\sigma_{ni}^2}{s_n^2} \leq 1.$$ 

Thus, for all $n > N$, we know that

$$\left| 1 - \frac{t^2 \sigma_{ni}^2}{2s_n^2} \right| \leq 1.$$
for all $i$. Since a characteristic function always has modulus bounded above by 1, we conclude from part (a) that

\[
\left| \prod_{i=1}^{n} \phi_{Y_{ni}} \left( \frac{t}{s_n} \right) - \prod_{i=1}^{n} \left( 1 - \frac{t^2 \sigma_{ni}^2}{2 s_n^2} \right) \right| \leq \sum_{i=1}^{n} \left| \phi_{Y_{ni}} \left( \frac{t}{s_n} \right) - \left( 1 - \frac{t^2 \sigma_{ni}^2}{2 s_n^2} \right) \right|.
\]

Now we may use part (b) to conclude that

\[
\left| \prod_{i=1}^{n} \phi_{Y_{ni}} \left( \frac{t}{s_n} \right) - \prod_{i=1}^{n} \left( 1 - \frac{t^2 \sigma_{ni}^2}{2 s_n^2} \right) \right| \leq \sum_{i=1}^{n} \left[ \frac{\epsilon \left| t \right|^3 \sigma_{ni}^2}{s_n^2} + \frac{t^2}{s_n} \mathbb{E} \left( Y_{ni}^2 I \{|Y_{ni}| \geq \epsilon s_n\} \right) \right]
\]

\[
= \epsilon \left| t \right|^3 + \frac{t^2}{s_n} \sum_{i=1}^{n} \mathbb{E} \left( Y_{ni}^2 I \{|Y_{ni}| \geq \epsilon s_n\} \right).
\]

(e) Use part (a) to prove that

\[
\left| \prod_{i=1}^{n} \left( 1 - \frac{t^2 \sigma_{ni}^2}{2 s_n^2} \right) - \prod_{i=1}^{n} \exp \left( -\frac{t^2 \sigma_{ni}^2}{2 s_n^2} \right) \right| \leq \frac{t^4}{4 s_n^4} \sum_{i=1}^{n} \sigma_{ni}^4 \leq \frac{t^4}{4 s_n^2} \max_{1 \leq i \leq n} \sigma_{ni}^2.
\]

**Hint:** Prove that for $x \leq 0$, $|1 + x - \exp(x)| \leq x^2$.

**Sketch of solution:** First, we prove that for $x \leq 0$, $|1 + x - \exp(x)| \leq x^2$. If we let $f(x) = 1 + x - \exp(x)$, then clearly $f(0) = 0$. Furthermore, $f'(x) = 1 - \exp(x)$, which is positive for all $x < 0$. Therefore, $f(x)$ is strictly increasing on $(-\infty, 0)$, which means that $f(x) < f(0) = 0$ for $x < 0$. We have just shown that the absolute value of $f(x)$ equals $-f(x)$ for all $x \leq 0$, which means it remains only to prove that $-f(x) \leq x^2$ for all $x \leq 0$.

Equivalently, we must prove that $f(x) + x^2 \geq 0$ for all $x \leq 0$. Here, we note that the derivative of $f(x) + x^2$ equals $f'(x) + 2x$, which is simply $f(x) + x$. We have already shown that $f(x) < 0$ for $x < 0$, so clearly $f(x) + x$ is also negative whenever $x$ is negative. This means that $f(x) + x^2$ is a strictly decreasing function of $x^2$ on $(-\infty, 0)$, so its value on this interval must always be larger than $f(0) + 0^2 = 0$, which is what was to be proven.

Using an argument similar to the one used in part (d), together with the fact that $|\exp(a)| \leq 1$ is always true for real $a \leq 0$, part (a) allows us to conclude that

\[
\left| \prod_{i=1}^{n} \left( 1 - \frac{t^2 \sigma_{ni}^2}{2 s_n^2} \right) - \prod_{i=1}^{n} \exp \left( -\frac{t^2 \sigma_{ni}^2}{2 s_n^2} \right) \right| \leq \sum_{i=1}^{n} \left| 1 - \frac{t^2 \sigma_{ni}^2}{2 s_n^2} - \exp \left( \frac{t^2 \sigma_{ni}^2}{2 s_n^2} \right) \right|
\]

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\[ \leq \sum_{i=1}^{n} \frac{t^4 \sigma^4_{n_i}}{4s^4_n}. \]

Now we may use the fact that \( \sigma^4_{n_i} \leq \sigma^2_{n_i} \max_{1 \leq j \leq n} \sigma^2_{n_j} \) to conclude that
\[
\sum_{i=1}^{n} \frac{t^4 \sigma^4_{n_i}}{4s^4_n} \leq \frac{t^4}{4s^2_n} \max_{1 \leq j \leq n} \sigma^2_{n_j} \sum_{i=1}^{n} \frac{\sigma^2_{n_i}}{s^2_n} = \frac{t^4}{4s^2_n} \max_{1 \leq j \leq n} \sigma^2_{n_j}.
\]

(f) Now put it all together. Show that
\[
\left| \prod_{i=1}^{n} \phi_{Y_{n_i}} \left( \frac{t}{s_n} \right) - \prod_{i=1}^{n} \exp \left( -\frac{t^2 \sigma^2_{n_i}}{2s^2_n} \right) \right| \to 0,
\]
proving (4.12).

**Sketch of solution:** By first using the triangle inequality and then combining the results of parts (d) and (e), we may conclude that for \( n > N \),
\[
\left| \prod_{i=1}^{n} \phi_{Y_{n_i}} \left( \frac{t}{s_n} \right) - \prod_{i=1}^{n} \exp \left( -\frac{t^2 \sigma^2_{n_i}}{2s^2_n} \right) \right| \leq \epsilon |t|^3 + \frac{t^2}{s^2_n} \sum_{i=1}^{n} \mathbb{E} \left( \frac{Y^2_{n_i} I \{ |Y_{n_i}| \geq \epsilon s_n \} }{s^2_n} \right) + \frac{t^4}{4s^2_n} \max_{1 \leq j \leq n} \sigma^2_{n_j}.
\]
The last two terms on the right go to zero as \( n \to \infty \) because of the Lindeberg condition (4.12) and condition (4.13), respectively. We conclude therefore that
\[
\limsup_{n} \left| \prod_{i=1}^{n} \phi_{Y_{n_i}} \left( \frac{t}{s_n} \right) - \prod_{i=1}^{n} \exp \left( -\frac{t^2 \sigma^2_{n_i}}{2s^2_n} \right) \right| \leq \epsilon |t|^3.
\]

But since \( \epsilon \) is arbitrary, we conclude that for any fixed \( t \),
\[
\left| \prod_{i=1}^{n} \phi_{Y_{n_i}} \left( \frac{t}{s_n} \right) - \prod_{i=1}^{n} \exp \left( -\frac{t^2 \sigma^2_{n_i}}{2s^2_n} \right) \right| \to 0. \tag{4.21}
\]

This actually proves the result (!!) because the two functions on the left hand side are the characteristic functions of \( T_n/s_n \) and a standard normal distribution, respectively.
Exercise 4.12  (a) Suppose that $X_1, X_2, \ldots$ are independent and identically distributed with $E X_i = \mu$ and $0 < \text{Var} X_i = \sigma^2 < \infty$. Let $a_{n1}, \ldots, a_{nn}$ be constants satisfying

$$\frac{\max_{i \leq n} a_{ni}^2}{\sum_{j=1}^n a_{nj}^2} \to 0 \text{ as } n \to \infty.$$ 

Let $T_n = \sum_{i=1}^n a_{ni} X_i$, and prove that $\frac{(T_n - E T_n)}{\sqrt{\text{Var} T_n}} \overset{d}{\to} N(0, 1)$.

**Sketch of solution:** It suffices to check the Lindeberg condition for the $a_{ni}(X_i - \mu)$. To this end, let $m_n = \max_{1 \leq i \leq n} a_{ni}^2$ and observe that for any $\epsilon > 0$,

$$I\{|a_{ni}(X_i - \mu)| \geq \epsilon s_n\} \leq I\{m_n(X_i - \mu)^2 \geq \epsilon^2 s_n^2\},$$

where as usual $s_n^2$ is the sum of the variances of the $a_{ni}X_i$. Furthermore, the random variables

$$Y_i \overset{\text{def}}{=} (X_i - \mu)^2 I\{m_n(X_i - \mu)^2 \geq \epsilon^2 s_n^2\}, 1 \leq i \leq n,$$

are i.i.d. Thus, we obtain

$$\frac{1}{s_n^2} \sum_{i=1}^n E[a_{ni}^2(X_i - \mu)^2 I\{|a_{ni}(X_i - \mu)| \geq \epsilon s_n\}] \leq \frac{1}{s_n^2} \sum_{i=1}^n a_{ni}^2 E Y_i \leq \frac{E Y_1}{\sigma^2}.$$ 

It only remains to show that $E Y_1 \to 0$, which follows from the dominated convergence theorem because $|Y_1| \leq (X_1 - \mu)^2$ and the fact that $Y_1 \overset{P}{\to} 0$ (the latter fact is because the indicator in the definition of $Y_i$ is identically zero with probability approaching one as $n \to \infty$ due to the fact that $s_n^2/m_n \to \infty$).

(b) Reconsider Example 2.22, the simple linear regression case in which

$$\hat{\beta}_{0n} = \sum_{i=1}^n v_i^{(n)} Y_i \text{ and } \hat{\beta}_{1n} = \sum_{i=1}^n w_i^{(n)} Y_i,$$

where

$$w_i^{(n)} = \frac{z_i - \bar{z}_n}{\sum_{j=1}^n (z_j - \bar{z}_n)^2} \text{ and } v_i^{(n)} = \frac{1}{n} - \bar{z}_n w_i^{(n)}$$

for constants $z_1, z_2, \ldots$. Using part (a), state and prove sufficient conditions on the constants $z_i$ that ensure the asymptotic normality of $\sqrt{n}(\hat{\beta}_{0n} - \beta_0)$ and $\sqrt{n}(\hat{\beta}_{1n} - \beta_1)$. You may assume the results of Example 2.22, where it was shown that $E \hat{\beta}_{0n} = \beta_0$ and $E \hat{\beta}_{1n} = \beta_1$. 212
Sketch of solution: To prove the desired results, we need to show two things: (i) \((\hat{\beta}_0 - \beta_1)/\sqrt{\text{Var} \hat{\beta}_0}\) is asymptotically normal, and (ii) \(\sqrt{n} \text{Var} \hat{\beta}_0\) converges to some nonzero constant (and similar results for \(\hat{\beta}_1\)). For (ii), we may verify that

\[
n \text{Var} \hat{\beta}_0 = \sigma^2 + \frac{n\sigma^2 z^2}{\sum_{j=1}^{n}(z_i - \bar{z}_n)^2} \quad \text{and} \quad n \text{Var} \hat{\beta}_1 = \frac{n\sigma^2}{\sum_{j=1}^{n}(z_i - \bar{z}_n)^2}.
\]

Thus, we conclude that (ii) occurs for \(\beta_1\) and \(\beta_0\) if both \(\frac{1}{n} \sum_{j=1}^{n}(z_i - \bar{z}_n)^2 \to \sigma^2 > 0\) and \(\bar{z}_n \to \mu_z\). For condition (i), it suffices to check that the condition of part (a) is satisfied for the \(v_i^{(n)}\) and the \(w_i^{(n)}\), since each \(Y_i\) is merely a shifted version of the corresponding \(\epsilon_i\) and the \(\epsilon_i\) are i.i.d. For the \(w_i^{(n)}\), a bit of algebra shows that the condition in part (a) is equivalent to

\[
\max_{1 \leq i \leq n}(z_i - \bar{z}_n)^2 \to 0.
\]

Notice that this implies Condition (2.18) and, in combination with \(\bar{z}_n \to \mu_z\), Condition (2.17), which were shown in Example 2.22 to be sufficient for the consistency of \(\hat{\beta}_{1n}\) and \(\hat{\beta}_{0n}\). Furthermore, since we’re already assuming \(\frac{1}{n} \sum_{j=1}^{n}(z_i - \bar{z}_n)^2 \to \sigma^2_2\) to satisfy (ii), we can can rewrite the condition in part (a) as

\[
\frac{\max_{1 \leq i \leq n}(z_i - \bar{z}_n)^2}{n} \to 0, \quad \text{or} \quad \max_{1 \leq i \leq n}(z_i - \bar{z}_n)^2 = o(n).
\]

Similarly, we may show that if \(\max_{1 \leq i \leq n}(z_i - \bar{z}_n)^2 = o(n)\), then

\[
\frac{\max_{1 \leq i \leq n}(v_i^{(n)})^2}{\sum_{j=1}^{n}(v_j^{(n)})^2} \to 0.
\]

(Can you verify this fact?)

Exercise 6.7 Let \(X_1, \ldots, X_n\) be independent uniform(0, 2\(\theta\)) random variables.

(a) Let \(M = (X_{(1)} + X_{(n)})/2\). Find the asymptotic distribution of \(n(M - \theta)\).

Sketch of solution: From Example 6.3, we know that

\[
\frac{n}{2\theta} \left( \begin{array}{c} X_{(1)} \\ 2\theta - X_{(n)} \end{array} \right) \xrightarrow{d} \left( \begin{array}{c} Y_1 \\ Y_2 \end{array} \right),
\]

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where $Y_1$ and $Y_2$ are independent standard exponential variables. This means that

$$n(M - \theta) = \frac{n}{2} [X_{(1)} - (2\theta - X_{(n)})] \overset{d}{\to} \theta(Y_1 - Y_2).$$

The right hand side above, $\theta(Y_1 - Y_2)$, is a double-exponential, or Laplace, distribution centered at zero and with variance $2\theta^2$.

(b) Compare the asymptotic performance of the three estimators $M$, $X_n$, and the sample median $\tilde{X}_n$ by considering their relative efficiencies.

**Sketch of solution:** From part (a), we may approximate the variance of $M$ by the asymptotically-inspired expression $2\theta^2/n^2$. The sample mean, on the other hand, has exactly variance $\theta^2/3n$ because each $X_i$ has variance $(2\theta)^2/12$. Finally, by Theorem 6.7, the sample median satisfies

$$\sqrt{n}(\tilde{X}_n - \theta) \overset{d}{\to} N\left(0, \frac{(2\theta^2)}{4}\right),$$

so as an asymptotic approximation we may write $\text{Var} \tilde{X}_n \approx \theta^2/n$. We conclude that the relative efficiency of $M$ relative to both $X_n$ and $\tilde{X}_n$ goes to infinity as $n \to \infty$, while the relative efficiency of $X_n$ relative to $\tilde{X}_n$ goes to 3.

(c) For $n \in \{101, 1001, 10001\}$, generate 500 samples of size $n$, taking $\theta = 1$. Keep track of $M$, $X_n$, and $\tilde{X}_n$ for each sample. Construct a $3 \times 3$ table in which you report the sample variance of each estimator for each value of $n$. Do your simulation results agree with your theoretical results in part (b)?

**Sketch of solution:** Below is some R code that summarizes the simulation results. They are very close to the asymptotic approximation in every case.

```r
f <- function(n) {
  x <- 2*runif(n)
  c(M=(min(x)+max(x))/2, Xbar=mean(x), Xtilde=median(x))
}
rbind(n101 = apply(replicate(500, f(101)), 1, var),
      n1001 = apply(replicate(500, f(1001)), 1, var),
      n10001 = apply(replicate(500, f(10001)), 1, var))
```
Exercise 6.10  Suppose $X_1, \ldots, X_n$ is a simple random sample from a distribution that is symmetric about $\theta$, which is to say that $P(X_i \leq x) = F(x - \theta)$, where $F(x)$ is the distribution function for a distribution that is symmetric about zero. We wish to estimate $\theta$ by $(Q_p + Q_{1-p})/2$, where $Q_p$ and $Q_{1-p}$ are the $p$ and $1-p$ sample quantiles, respectively. Find the smallest possible asymptotic variance for the estimator and the $p$ for which it is achieved for each of the following forms of $F(x)$:

(a) Standard Cauchy

(b) Standard normal

(c) Standard double exponential

**Hint:** For at least one of the three parts of this question, you will have to solve for a minimizer numerically.

**Sketch of solution:** In the case of a distribution that is symmetric about $\theta$, we know that the derivative of the cdf at the true $p$th quantile $\xi_p$ is the same as at $\xi_{1-p}$. Therefore, Theorem 6.7 says that

$$\sqrt{n} \left\{ \left( \frac{Q_p}{Q_{1-p}} \right) - \left( \frac{\xi_p}{\xi_{1-p}} \right) \right\} \xrightarrow{d} N \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{p}{F'(\xi_p)^2} \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} \right\},$$

where $p$ is assumed to be $\leq 1/2$. Furthermore, since $\theta = (\xi_p + x_{1-p})/2$, we conclude that

$$\sqrt{n} \left( \frac{Q_p - Q_{1-p}}{2} - \theta \right) \xrightarrow{d} \frac{p}{2F'(\xi_p)^2},$$

which means that $p/f(\xi_p)^2$ is the function we wish to minimize, where $f(x)$ is the density function in question. For part (c), the function to minimize is $p/p^2 = 1/p$, which is minimized on $(0, 1/2]$ at $p = 1/2$. The minimizing values of $p$ for parts (a) and (b) are given by the R code below.

```r
optimize(function(p) p/(dcauchy(qcauchy(p)))^2, interval=c(0,.5))$min
[1] 0.4435034
optimize(function(p) p/(dnorm(qnorm(p)))^2, interval=c(0,.5))$min
[1] 0.2702512
```