Problem 1. Let $X_1, X_2, \ldots$ be iid Poisson($\theta$) random variables. For each $i$, let $Y_i = I\{X_i = 0\}$. Find the asymptotic distribution of $2X_n + \log \sum_{i=1}^{n} Y_i$. Show all steps, including finding the joint asymptotic distribution of $X_n$ and $Y_n$.

A Poisson($\theta$) random variable $X$ has probability mass function $P_{\theta}(X = k) = \frac{\theta^k e^{-\theta}}{k!}$ for $k = 0, 1, 2, \ldots$. Furthermore, $E_{\theta} X = \text{Var}_{\theta} X = \theta$.

Problem 2. For $\theta > 0$, let $X_1, \ldots, X_n$ be iid from an exponential distribution with cdf $F_X(x) = 1 - e^{-x/\theta}$ and density $f_X(x) = \frac{1}{\theta} e^{-x/\theta}$ for all $x > 0$. This implies $E(X_i) = \theta$ and $\text{Var}(X_i) = \theta^2$.

(a) Find the asymptotic distribution of the midquartile range $R_n = \frac{Q_{.25} + Q_{.75}}{2}$, where $Q_{.25}$ and $Q_{.75}$ are the .25 and .75 quantiles (Note: Because this distribution is skewed, $R_n$ is consistent for $\theta \log(4/\sqrt{3})$, not $\theta$).

Compare $R_n/\log(4/\sqrt{3})$ to $\bar{X}_n$ as an estimator of $\theta$ by comparing their asymptotic variances.

(b) Find the asymptotic distribution of the range, defined as the difference between the largest and smallest observations.

Problem 3. Let $X_1, X_2, \ldots$ be independent and identically distributed Gumbel random variables with cumulative distribution function $F_X(x) = \exp\{-e^{-x/\theta}\}$.

Derive the asymptotic distribution of $X_{(n)} - \theta \log n$, where $X_{(n)}$ is the largest of $X_1, \ldots, X_n$.

Problem 4. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed exponential random variables with mean $\theta$.

The exponential distribution has distribution function $F_X(x) = 1 - \exp\{-x/\theta\}$ and density $f_X(x) = \frac{1}{\theta} \exp\{-x/\theta\}$. It satisfies $E X^k = k! \theta^k$ for positive integers $k$.

(a) For arbitrary $0 < p < 1$, let $Q(p, n)$ denote the $p$th sample quantile from a sample of size $n$. Find a function $k(p)$ such that $k(p) Q(p, n) \xrightarrow{p} \theta$.

Also, find the asymptotic distribution of $k(p) Q(p, n)$. Show your work.

(b) Define

$$s_n^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2.$$ 

Find the asymptotic distribution of $s_n^2$.

Problem 5. Suppose $X_1, X_2, \ldots$ are independent with $X_i \sim \text{Beta}(\alpha_i, \alpha_i)$, where $0 < \alpha_i < 2$. Prove that

$$\sum_{i=1}^{n} \frac{(X_i - \frac{1}{2})}{\sqrt{s_n^2}} \xrightarrow{d} N(0, 1),$$
where \( s_n^2 = \sum_{i=1}^{n} \text{Var}(X_i) \), by verifying the Lindeberg condition or the Lyapunov condition.

The Beta(\( \alpha, \beta \)) distribution has expectation \( \frac{\alpha}{\alpha + \beta} \), variance \( \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} \), and density \( \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I\{0 < x < 1\} \).

**Problem 6.** Suppose that we flip a fair coin 1000 times and count the number of runs of heads. Let this number be \( R \). A run of heads is a sequence of one or more heads that is preceded and followed by tails (we may imagine the existence of a 0th and 1001th toss that are both tails).

For example, there are 3 runs of heads in the sequence T T H H H H T H H T H.

Recall that \( R \) may be expressed as \( \sum_{i=1}^{1000} Y_i \), where \( Y_i \) is the indicator that a run of heads begins on the \( i \)th flip.

Find an approximate distribution for \( R \) based on an asymptotic result.

**Problem 7.** Suppose that for all \( n \), \( X_1, \ldots, X_{n} \) are independent and identically distributed as Uniform(\( -n, n \)). Let \( X_n = \frac{1}{n} \sum_{k=1}^{n} X_{nk} \). Prove that

\[
\frac{X_n}{\sqrt{n}} \xrightarrow{d} N(0, \tau^2)
\]

for some constant \( \tau^2 \) and find the value of \( \tau^2 \).

A uniform(\( a, b \)) random variable has density function \( I\{a < x < b\} / (b - a) \), mean \( (b + a)/2 \), and variance \( (b - a)^2 / 12 \).

**Problem 8.** Suppose \( X_1, X_2, \ldots \) are iid Poisson(\( \theta \)) random variables. Find the asymptotic distribution of \( (S_n - E S_n) / \sqrt{\text{Var} S_n} \), where

\[
S_n = \sum_{i=2}^{n+1} X_i I\{X_{i-1} = 0\}.
\]

The Poisson(\( \theta \)) distribution has expectation \( \theta \), variance \( \theta \), and mass function \( p(x) = e^{-\theta} \theta^x / x! \) for \( x \) a nonnegative integer.

**Problem 9.** Suppose that \( X_1, \ldots, X_{n} \) are iid with

\[
P(X_i = 0) = \theta \quad \text{and} \quad P\left( X_i = -\sqrt{1-\theta} \right) = P\left( X_i = \sqrt{1-\theta} \right) = \frac{1-\theta}{2}.
\]

Define \( Y_i = I\{X_i = 0\} \).

Let \( \mathbf{Z}^{(i)} = (X_i, Y_i) \). Find a function \( g(x, y) = [g_1(x, y), g_2(x, y)] \) that is a variance-stabilizing transformation in the sense that the asymptotic distribution of \( g[\mathbf{Z}] - g[E(\mathbf{Z}^{(i)})] \) has a covariance matrix that doesn’t depend on \( \theta \), where \( \mathbf{Z} \) is the sample mean of the \( \mathbf{Z}^{(i)} \).

You may find it helpful to know that

\[
\frac{d}{dt} 2\sin^{-1}(\sqrt{t}) = \frac{1}{\sqrt{t(1-t)}}.
\]
Problem 10. Suppose \( X_1, X_2, \ldots \) are independent with \( X_i \sim \text{Beta}(\alpha_i, \alpha_i) \), where \( 0 < \alpha_i < 2 \). Prove that
\[
\frac{\sum_{i=1}^{n} (X_i - \frac{1}{2})}{\sqrt{s_n^2}} \overset{d}{\to} N(0, 1),
\]
where \( s_n^2 = \sum_{i=1}^{n} \text{Var}(X_i) \), by verifying the Lindeberg condition or the Lyapunov condition.

The Beta(\( \alpha, \beta \)) distribution has expectation \( \frac{\alpha}{\alpha + \beta} \), variance \( \frac{\alpha \beta}{(1 + \alpha + \beta)(\alpha + \beta)^2} \), and density
\[
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} I\{0 < x < 1\}.
\]

Problem 11. Let \( X_1, \ldots, X_n \) be an iid sample from Poisson(\( \theta \)). Throughout this problem, you may assume that the Poisson distribution satisfies all relevant regularity conditions.

(a) Show that the Jeffreys prior on \((0, \infty)\) is the improper prior density \( \lambda(\theta) = \frac{1}{\sqrt{\theta}} \). To do this, it suffices to show that \( \lambda(\theta) \) is proportional to \( \sqrt{I(\theta)} \).

The Poisson(\( \theta \)) distribution has expectation \( \theta \), variance \( \theta \), and mass function \( p(x) = e^{-\theta} \theta^x / x! \) for \( x \) a nonnegative integer.

(b) Show that with the improper Jeffreys prior \( \lambda(\theta) = \frac{1}{\sqrt{\theta}} \), the posterior distribution of \( \theta \) is gamma. Find the Bayes estimator \( \delta_n = \mathbb{E}(\theta | X_1, \ldots, X_n) \) using this prior. Finally, give the asymptotic distribution of \( \sqrt{n}(\delta_n - \theta) \).

The Gamma(\( \alpha, \beta \)) distribution has expectation \( \frac{\alpha}{\beta} \), variance \( \frac{\alpha}{\beta^2} \), and density \( \beta^{\alpha} x^{\alpha-1} e^{-\beta x} I\{x > 0\} / \Gamma(\alpha) \).

Problem 12. Suppose that for some \( p \in (0, 1) \),
\[
X_1, X_2, \ldots \overset{iid}{\sim} \text{Bernoulli} (p) \\
Y_1, Y_2, \ldots \overset{iid}{\sim} \text{Uniform} (0, 1)
\]
and the \( X \)'s are independent of the \( Y \)'s. Let \( Z_i = Y_i - X_i \) and define \( M_n = \max_{1 \leq i \leq n} \{Z_i\} \). Find the asymptotic distribution of \( n(1 - M_n) \).

Problem 13. Suppose that for \( \theta > 2 \), \( X_1, X_2, \ldots, X_n \) is an iid sample from a Pareto distribution with density function \( f_\theta(x) = \frac{\theta}{x^b+1} \), \( x > 1 \). This implies that
\[
\mathbb{E} X_i = \frac{\theta}{\theta - 1} \quad \text{and} \quad \text{Var} X_i = \frac{\theta}{(\theta - 2)(\theta - 1)^2}.
\]
Let \( \hat{\theta}_n \) denote the method of moments estimator of \( \theta \) found by setting \( \mathbb{E} X_i = \bar{X}_n \) and solving for \( \theta \).

(a) Find the asymptotic distribution of \( \hat{\theta}_n \).

(b) Is \( \hat{\theta}_n \) efficient? Support your answer.

(c) Let \( \delta_n \) denote the one-step Newton estimator based on \( \hat{\theta}_n \). (That is, \( \delta_n \) is the estimator given by a single step of Newton’s method for solving the likelihood equation, starting at \( \hat{\theta}_n \).) Find the explicit form of \( \delta_n \) and give its asymptotic distribution.