Problem 1. Let $X_1, X_2, \ldots$ be iid Poisson($\lambda$) random variables. For each $i$, let $Y_i = I\{X_i = 0\}$.

(a) [3 points] Find the joint asymptotic distribution of $X_n$ and $Y_n$.

(b) [2 points] Based on your answer to part (a), find the asymptotic distribution of $(X_n - \log Y_n)/2$.

(c) [2 points] Suppose we wish to test $H_0: \lambda = 1$ against $H_1: \lambda > 1$. Consider the test that rejects $H_0$ when a certain test statistic $T_n$ satisfies

$$T_n \geq 1 + \frac{1.645\sigma(1)}{\sqrt{n}},$$

where $\sigma^2(\lambda)$ is the asymptotic variance of $\sqrt{n}(T_n - \lambda)$. (Note: $1.645 = u_{.95}$.) If test A takes $T_n = \overline{X}_n$ and test B takes $T_n = -\log \overline{Y}_n$, find the asymptotic relative efficiency of test A to test B.

[You may assume without proof that under the alternatives $\lambda_n$, $\sqrt{n}(T_n - \lambda_n) \overset{L}{\to} N(0, \sigma^2(1))$.]

Problem 2. [2 points] Suppose $X^{(n)}$ is a random $k$-vector distributed as Multinomial$(n, p)$. Find a function $g: \mathbb{R} \to \mathbb{R}$ such that

$$n \sum_{i=1}^k \left[ g\left( \frac{X_{i}^{(n)}}{n} \right) - g(p_i) \right]^2 \overset{L}{\to} \chi^2_{k-1}.$$

In other words, the sequence above should have the same asymptotic distribution as $Y'Y$, where $Y = \sqrt{n}D[(X^{(n)}) - p]$ and $D = \text{diag}(1/\sqrt{p_1}, \ldots, 1/\sqrt{p_k})$.

Problem 3. For $\theta > 0$, let $X_1, \ldots, X_n$ be iid from an exponential distribution with cdf $F(x) = 1 - e^{-x/\theta}$ and density $f(x) = e^{-x/\theta}/\theta$ for all $x > 0$. This implies $E(X_i) = \theta$ and $\text{Var}(X_i) = \theta^2$.

(a) [3 points] Find the asymptotic distribution of the midquartile range $R_n = (Q_{.25} + Q_{.75})/2$, where $Q_{.25}$ and $Q_{.75}$ are the .25 and .75 quantiles (Note: Because this distribution is skewed, $R_n$ is consistent for $\theta \log(4/\sqrt{3})$, not $\theta$).

Compare $R_n/\log(4/\sqrt{3})$ to $\overline{X}_n$ as an estimator of $\theta$ by comparing their asymptotic variances.

(b) [2 points] Suppose we have a test that rejects $H_0: \theta = 1$ in favor of $H_1: \theta > 1$ whenever

$$\overline{X}_n \geq 1 + \frac{u_{.95}}{\sqrt{n}}.$$
Let \( \theta_n = (1 + \sqrt{n})^2/n \). If \( \beta_n(\theta_n) \) denotes the power of this test against the alternative \( \theta_n \), find the value of \( \lim_{n \to \infty} \beta_n(\theta_n) \). You may leave your answer in terms of the standard normal cdf \( \Phi(x) \). You may also assume without proof that if \( \theta_n \to 1 \), then
\[
\sqrt{n}(X_n - \theta_n) \xrightarrow{d} N(0, 1)
\]
under the alternatives \( \{\theta_n\} \).

(c) [1 point] Find the asymptotic distribution of the range, defined as the difference between the largest and smallest observations.