Problem 1. Let $X_1, X_2, \ldots$ be iid from a distribution with $F(x) = 1 + x^5$ for $x \in (-1, 0)$. Let $Y_n = \max_{1 \leq i \leq n} X_i$.

(a) Prove that $Y_n \xrightarrow{P} 0$.

(b) If $Y$ is a random variable for which $n^{1/5} Y \xrightarrow{d} Y$, find the cumulative distribution function of $Y$. Be careful to specify the range of values for which the cdf is valid.

Problem 2. For $0 < p < 1$ and $q = 1 - p$, let $X_1, X_2, \ldots$ be iid random variables such that

\begin{align*}
P(X_k = 0) &= P(X_k = 2p) = p/2, \\
P(X_k = p) &= q.
\end{align*}

(a) Define

$$\delta_n = \frac{1}{n} \sum_{k=1}^n X_k \left[ X_k^2 - 3(\bar{X}_n)^3 \right].$$

Show that $\delta_n$ is a consistent estimator for $p^3 = \text{Var}(X_k)$.

(b) Find a function $f(x)$ that doesn’t depend on $p$ such that

$$\sqrt{n} \left[ f(\bar{X}_n) - f(p) \right] \xrightarrow{d} N(0, 1).$$

(c) Let $Y_k = \sqrt{k}(X_k - p)$. If $s_n^2 = \sum_{k=1}^n \text{Var}(Y_k)$, prove that

$$\frac{\sum_{k=1}^n Y_k}{s_n} \xrightarrow{d} N(0, 1).$$

Problem 3. Let $X_1, X_2, \ldots$ be iid Poisson ($\lambda$) random variables. [That is, $P(X_k = m) = e^{-\lambda} \lambda^m / m!$ for $m = 0, 1, 2, \ldots$ and $E(X_k) = \text{Var}(X_k) = \lambda$.] Let $Z_k = I\{X_k = 0\}$ and $Y_k = I\{X_k \leq 1\}$ for all $k$.

(a) Find the asymptotic distribution of

$$\sqrt{n}(\lambda + \log Z_n).$$

(b) Prove that $(\bar{Y}_n/Z_n) - 1$ is a consistent estimator of $\lambda$.

Problem 4. Suppose $\lambda_1, \lambda_2, \ldots$ is a sequence of positive real numbers with $\lambda_n \to \lambda_0 > 0$. For a given $n$, let $X_{n1}, \ldots, X_{nn}$ be iid Poisson($\lambda_n$) random variables. Prove that

$$\sqrt{n} \left( \bar{X}_n - \lambda_n \right) \xrightarrow{d} N(0, \lambda_0),$$

where $\bar{X}_n = (\sum_{k=1}^n X_{nk})/n$. 

This closed-book midterm is worth 15 points. You have 75 minutes. Properties of some distributions are included in italics where appropriate.

**Problem 1.**[3 points] Suppose $X_0, X_1, \ldots$ are iid Poisson($\theta$) random variables. Define $Y_k = X_k I\{X_{k-1} = 0\}$ for $k = 1, 2, \ldots$. Find $E Y_n$ and the limit of $\text{Var}(\sqrt{n}Y_n)$, and then give the asymptotic distribution of $\sqrt{n}(Y_n - E Y_n)$.

The Poisson($\theta$) distribution has expectation $\theta$, variance $\theta$, and mass function $p(x) = e^{-\theta}\theta^x/x!$ for $x$ a nonnegative integer.

**Problem 2.**[3 points] Suppose that $X_1, X_2, \ldots$ are iid with $P(X_i = \theta) = \theta$ and $P(X_i = 2\theta - 1) = P(X_i = 1) = 1 - \theta$ for some $\theta \in (0, 1)$. Find a variance-stabilizing transformation $f(t)$ and a random variable $Y$ such that $\sqrt{n}\{f(X_n) - f(\theta)\} \overset{L}{\to} Y$ and the distribution of $Y$ does not depend on $\theta$.

**Problem 3.** Either construct an example of each of the following or prove that no such example exists:

(a) [2 points] A sequence $X_1, X_2, \ldots$ of random variables, a random variable $X$, and a constant $x_0$ such that $X_n \overset{L}{\to} X$ and $F_n(x_0) \neq F(x_0)$.

(b) [2 points] A sequence $X_1, X_2, \ldots$ of random variables such that $X_n \overset{P}{\to} 0$ and $E(X_n) \to 1$.

(c) [2 points] A sequence $X_1, X_2, \ldots$ of random variables, each with mean zero, such that $\text{Var} X_n \to 0$ and $X_n \overset{P}{\to} 1$.

**Problem 4.**[3 points] Suppose $X_1, X_2, \ldots$ are independent with $X_i \sim \text{Beta}(\alpha_i, \alpha_i)$, where $0 < \alpha_i < 2$. Prove that

$$\frac{\sum_{i=1}^n \left( X_i - \frac{1}{2} \right)}{\sqrt{s_n^2}} \overset{L}{\to} N(0, 1),$$

where $s_n^2 = \sum_{i=1}^n \text{Var}(X_i)$, by verifying the Lindeberg condition or the Lyapunov condition.

The Beta($\alpha, \beta$) distribution has expectation $\alpha/(\alpha + \beta)$, variance $\alpha\beta/[(1 + \alpha + \beta)(\alpha + \beta)^2]$, and density $\Gamma(\alpha + \beta)x^{\alpha-1}(1-x)^{\beta-1}I\{0 < x < 1\}/[\Gamma(\alpha)\Gamma(\beta)]$. 