Problem 1. [3 points] Let $X_1, \ldots, X_n$ be an iid sequence with $E X_i = \mu$ and $\text{Var} X_i = \sigma^2 < \infty$. Prove the central limit theorem: $\sqrt{n}(X_n - \mu) \overset{L}{\to} N(0, \sigma^2)$.

You may use the fact that the characteristic function of the standard normal distribution is $\exp\{-t^2/2\}$.

You may also use the fact that

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \cdots + \frac{(z - z_0)^r}{r!} \left\{ f^{(r)}(z_0) + o(1) \right\}$$

for a complex $f(z)$ with at least $r$ derivatives.

**SOLUTION:**

There was an issue here of how much you were allowed to assume. Several people used the Lindeberg-Feller theorem to “prove” this, and technically I didn’t exclude this possibility although I thought that my mention of characteristic functions and Taylor’s theorem should have tipped everyone off that I expected a direct proof. I should have been more specific!

Letting $Z_n = \sqrt{n}(X_n - \mu)/\sigma$ and $Y_i = (X_i - \mu)/(\sigma \sqrt{n})$, then $Z_n = \sum_{i=1}^n Y_i$ and

$$\phi_{Z_n}(t) = \left\{ \phi_{Y_1}(t) \right\}^n = \left[ E \left\{ 1 + iY_1t - \frac{Y_1^2t^2}{2}(1 + o(1)) \right\} \right]^n.$$

Since $E Y_1 = 0$ and $E Y_1^2 = \frac{1}{n}$, we conclude that

$$\phi_{Z_n}(t) = \left[ 1 - \frac{t^2(1 + o(1))}{2n} \right]^n \to \exp \left\{ -\frac{t^2}{2} \right\}$$

and thus $Z_n \overset{L}{\to} N(0, 1)$ by the continuity theorem.
Problem 2. [3 points] Suppose that for some \( p \in (0, 1) \),
\[
X_1, X_2, \ldots \overset{iid}{\sim} \text{Bernoulli } (p) \\
Y_1, Y_2, \ldots \overset{iid}{\sim} \text{Uniform } (0, 1)
\]
and the \( X \)'s are independent of the \( Y \)'s. Let \( Z_i = Y_i - X_i \) and define \( M_n = \max_{1 \leq i \leq n} \{ Z_i \} \). Find the asymptotic distribution of \( n(1 - M_n) \).

SOLUTION:
For \( 0 < u < 1 \),
\[
P(Z_i < u) = \text{E}(Z_i < u|X_i) = p + (1 - p)u
\]
Thus, for \( t > 0 \),
\[
P(n(1 - M_n) > t) = P\left( M_n < 1 - \frac{t}{n}\right) = \left\{ p + (1 - p) \left( 1 - \frac{t}{n}\right) \right\}^n \rightarrow \exp(-t(1 - p)).
\]
Thus, \( n(1 - M_n) \xrightarrow{L} \text{Exponential} \{1/(1-p)\} \).

Problem 3. [4 points] Let \( X_1, \ldots, X_n \) be an iid sample from the distribution \( F \). Suppose that \( E_F X = 0 \), \( \text{Var}_F X = \sigma^2 \), and \( \text{Var}_F X^2 = \tau^2 < \infty \). Let \( \theta = E_F \phi(X_1, X_2) \), where \( \phi(a,b) = (a - b)^2 \). Let \( U_n \) denote the U-statistic for estimating \( \theta \).

(a) Give the asymptotic distribution for \( U_n \).

SOLUTION: We have
\[
\phi_1(X) = E \left( X^2 - 2XX_2 + X^2_2 |X \right) = X^2 - 2X \text{E} X_2 + \text{E} X^2_2 = X^2 + \sigma^2.
\]
Thus, \( \sigma^2_1 = \text{Var}_F X^2 = \tau^2 \) and since \( a = 2 \) we conclude \( \sqrt{n}(U_n - \theta) \xrightarrow{L} N(0, 4\tau^2) \).

(b) Show that \( U_n = ks_n \), where \( s_n \) is the usual unbiased estimator of \( \sigma^2 \) and \( k \) is some constant.

SOLUTION:
\[
U_n = \frac{2}{n(n - 1)} \sum_{i<j} (X_i^2 - 2X_iX_j + X_j^2)
\]
\[
= \frac{2}{n(n - 1)} \left( \sum_i (n - i)X_i^2 + \sum_j (j - 1)X_j^2 - 2 \sum_{i<j} X_iX_j \right)
\]
\[
= \frac{2}{n(n - 1)} \left( n \sum_i X_i^2 - \sum_i X_i^2 - 2 \sum_{i<j} X_iX_j \right)
\]
\[
= \frac{2}{n - 1} \left\{ \sum_i X_i^2 - \frac{1}{n} \left( \sum_i X_i \right)^2 \right\}
\]
\[
= 2s^2_n.
\]
Problem 4. [6 points] Suppose that for $\theta > 2$, $X_1, \ldots, X_n$ is an iid sample from a Pareto distribution with density function $f_\theta(x) = \frac{\theta}{x^{\theta+1}}$, $x > 1$. This implies that

$$E X_i = \frac{\theta}{\theta-1} \quad \text{and} \quad Var X_i = \frac{\theta}{(\theta-2)(\theta-1)^2}. $$

Let $\hat{\theta}_n$ denote the method of moments estimator of $\theta$ found by setting $E X_i = \bar{X}_n$ and solving for $\theta$.

(a) Find the asymptotic distribution of $\hat{\theta}_n$.

**SOLUTION:** Note that $\hat{\theta}_n = \bar{X}_n/(\bar{X}_n - 1)$. By the CLT,

$$\sqrt{n} \left( \bar{X}_n - \frac{\theta}{\theta-1} \right) \xrightarrow{L} N \left( 0, \frac{\theta}{(\theta-2)(\theta-1)^2} \right).$$

Letting $g(x) = x/(x-1)$, we obtain $g' \{\theta/(\theta-1)\} = (\theta-1)^2$. Therefore, the delta method gives

$$\sqrt{n} \left( \hat{\theta}_n - \theta \right) \xrightarrow{L} N \left( 0, \frac{\theta(\theta-1)^2}{\theta-2} \right).$$

(b) Is $\hat{\theta}_n$ efficient? Support your answer.

**SOLUTION:** Since

$$\frac{d}{d\theta} \log f_\theta(x) = -\frac{1}{\theta^2},$$

we know $I(\theta) = 1/\theta^2$ and thus an efficient estimator $\hat{\theta}_n$ would satisfy $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{L} N(0, \theta^2)$. From part (a), we see that $\hat{\theta}_n$ is not efficient.

(c) Let $\delta_n$ denote the one-step Newton estimator based on $\hat{\theta}_n$. (That is, $\delta_n$ is the estimator given by a single step of Newton’s method for solving the likelihood equation, starting at $\hat{\theta}_n$.) Find the explicit form of $\delta_n$ and give its asymptotic distribution.

**SOLUTION:** With $\ell(\theta) = n \log(\theta) - (\theta + 1) \sum_i \log X_i$, we obtain $\ell'(\theta) = \frac{n}{\theta} - \sum_i \log X_i$ and $\ell''(\theta) = -n/\theta^2$. Therefore,

$$\delta_n = \hat{\theta}_n - \frac{\ell'(\hat{\theta}_n)}{\ell''(\hat{\theta}_n)} = \frac{2\hat{\theta}_n - \theta^2 \sum_i \log X_i}{n}.$$

Because we know that one-step Newton estimators are efficient under regularity conditions (which are clearly satisfied here), we conclude that

$$\sqrt{n}(\delta_n - \theta) \xrightarrow{L} N(0, \theta^2).$$
Problem 5. [4 points] For $\theta > 0$, suppose $X_1, X_2, \ldots$ are iid from an exponential distribution with mean $\theta$. This implies that $\text{Var } X_i = \theta^2$ and $E(X_i - \theta)^3 = 2\theta^3$. We wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$.

(a) Suppose that $\theta_n > \theta_0$ for all $n$ and $\theta_n \to \theta_0$. Prove that under the alternatives $\{\theta_n\}$,

$$\frac{\sqrt{n}(X_n - \theta_n)}{\theta_0} \overset{L}{\to} N(0, 1).$$

**SOLUTION:** I will use the additional fact that $E(X_i - \theta)^4 = 9\theta^4$. Let $Y_{ni} = X_i - \theta_n$ under alternative $\theta_n$. Then because

$$\frac{1}{(n\theta_n^2)^{1/2}} \sum_{i=1}^{n} E |Y_{ni}|^4 = \frac{9n\theta^4}{n^{2}\theta_n^4} \to 0$$

as $n \to \infty$, the Lyapunov condition is satisfied and therefore

$$\frac{\sum_{i=1}^{n} Y_{ni}}{\sqrt{n}\theta_n} \overset{L}{\to} N(0, 1).$$

Since $\theta_n/\theta_0 \to 1$, Slutsky’s theorem implies that

$$\frac{\sum_{i=1}^{n} (X_i - \theta_n)}{\sqrt{n}\theta_0} \overset{L}{\to} N(0, 1)$$

as desired.

(b) Suppose we reject $H_0$ whenever

$$\frac{\bar{X}_n}{\theta_0} > 1 + \frac{1.645}{\sqrt{n}}.$$  

(Note: 1.645 is the 95th percentile of the standard normal distribution.) If $\beta_n(\theta_n)$ denotes the power of this test against the alternative $\theta_n$, evaluate

$$\lim_{n \to \infty} \beta_n \left( \theta_0 + \frac{2(1.645)}{\sqrt{n}} + \frac{1.645}{n} \right).$$

**SOLUTION:** Letting $\theta_n = \theta_0 + \frac{2(1.645)}{\sqrt{n}} + \frac{1.645}{n}$, we see that $\sqrt{n}(\theta_n - \theta_0) \to 2(1.645)$ as $n \to \infty$. With $\mu(\theta) = \tau(\theta) = \theta$, we obtain $\mu'(\theta) = 1$ and thus

$$\beta_n(\theta_n) \to \Phi \left( \frac{2(1.645)(1)}{\theta_0} - 1.645 \right).$$

I made a bit of a mistake when I wrote this problem: I had wanted the answer to come out to be $\Phi(1.645)$ or 0.95, but to get this I should have asked for

$$\lim_{n \to \infty} \beta_n \left( \theta_0 \left( 1 + \frac{2(1.645)}{\sqrt{n}} + \frac{1.645}{n} \right) \right)$$

instead. Oh well, the problem still works in its current form.