Problem 1.  If $c$ is a constant such that $X_n \overset{L}{\rightarrow} c$, prove that $X_n \overset{P}{\rightarrow} c$.

Problem 2.  Suppose $X_1, X_2, \ldots$ are iid Bernoulli($p$) random variables.  Let $S_n = \sum_{i=2}^{n} X_i X_{i-1}$. Find (with justification) the asymptotic distribution of $S_n/n$.

Problem 3.  Either construct an example of each of the following or prove that no such example exists:

(a) [1 point] A sequence $X_1, X_2, \ldots$ of random variables, a random variable $X$, and a constant $x_0$ such that $X_n \overset{L}{\rightarrow} X$ and $F_n(x_0) \not\rightarrow F(x_0)$.

(b) [1 point] A sequence $X_1, X_2, \ldots$ of random variables such that $X_n \overset{P}{\rightarrow} 0$ and $E(X_n) \rightarrow 1$.

Problem 4.  Let $X_1, \ldots, X_n$ be iid from a continuous symmetric distribution centered at 0.  Suppose $(Y_1, \ldots, Y_n)$ is a permutation of $(X_1, \ldots, X_n)$ satisfying $|Y_1| < |Y_2| < \cdots < |Y_n|$; that is, the $Y_i$ are the $X_i$ arranged in order of increasing absolute value.

Let $W_n = \sum_{i=1}^{n} i I\{Y_i > 0\}$ be the usual signed-rank statistic.  Derive the asymptotic distribution of $W_n$, justifying your steps.

[We have seen at least two ways to do this.  You may of course use any valid method you choose.]

Problem 5.  Suppose $X_1, \ldots, X_n$ are iid random variables with cdf $F(x)$.  Let $\tilde{X}_n$ denote the sample median.  Suppose we wish to estimate $h(F) = \text{Var}(\tilde{X}_n) < \infty$.  We use a bootstrap scheme in which we draw $B$ random samples of size $n$ from $\hat{F}_n$, the empirical cdf, and let $M_i$ be the sample median of the $i$th sample, $i = 1, \ldots, B$.

If $\overline{M}_B$ denotes $(1/B) \sum_{i=1}^{B} M_i$, explain (with justification) what happens to

$$\frac{1}{B} \sum_{i=1}^{B} (M_i - \overline{M}_B)^2$$

as $B \rightarrow \infty$.

Problem 6.  Suppose $X_1, X_2, \ldots$ are iid Poisson($\theta$) random variables.  Find the asymptotic distribution of $(S_n - E S_n)/\sqrt{\text{Var} S_n}$, where

$$S_n = \sum_{i=2}^{n+1} X_i I\{X_{i-1} = 0\}.$$

The Poisson($\theta$) distribution has expectation $\theta$, variance $\theta$, and mass function $p(x) = e^{-\theta} \theta^x / x!$ for $x$ a nonnegative integer.
### Problem 7
Suppose that \( X_1, \ldots, X_n \) are iid with
\[
P(X_i = 0) = \theta \quad \text{and} \quad P \left( X_i = -\sqrt{1 - \theta} \right) = P \left( X_i = \sqrt{1 - \theta} \right) = \frac{1 - \theta}{2}.
\]

Define \( Y_i = I \{ X_i = 0 \} \).

Let \( Z(i) = (X_i, Y_i) \). Find a function \( g(x, y) = [g_1(x, y), g_2(x, y)] \) that is a variance-stabilizing transformation in the sense that the asymptotic distribution of \( g \left( \overline{Z} \right) - g \left( \text{E}(Z(i)) \right) \) has a covariance matrix that doesn’t depend on \( \theta \), where \( \overline{Z} \) is the sample mean of the \( Z(i) \).

You may find it helpful to know that
\[
\frac{d}{dt} 2 \sin^{-1}(\sqrt{t}) = \frac{1}{\sqrt{t(1-t)}}.
\]

### Problem 8
Let \( X_1, \ldots, X_n \) be an iid sample from Beta(\( \alpha, 1 \)); that is, \( f_\alpha (x) = \alpha x^{\alpha - 1} \). You may assume without proof that all relevant regularity conditions apply to the beta distribution.

The Beta(\( \alpha, \beta \)) distribution has expectation \( \alpha / (\alpha + \beta) \), variance \( \alpha \beta / [(1 + \alpha + \beta)(\alpha + \beta)^2] \), and density \( \Gamma(\alpha + \beta)x^{\alpha - 1}(1-x)^{\beta - 1}I \{ 0 < x < 1 \} / \Gamma(\alpha)\Gamma(\beta) \).

(a) Compute a Wald test statistic \( W_n \) that is asymptotically standard normal under \( H_0: \alpha = \alpha_0 \).

(b) Compute a Rao score test statistic \( R_n \) that is asymptotically standard normal under \( H_0: \alpha = \alpha_0 \).

### Problem 9
Suppose \( X_1, X_2, \ldots \) are independent with \( X_i \sim \text{Beta}(\alpha_i, \alpha_i) \), where \( 0 < \alpha_i < 2 \). Prove that
\[
\frac{\sum_{i=1}^{n} (X_i - \frac{1}{2})}{\sqrt{s_n^2}} \xi N(0, 1),
\]
where \( s_n^2 = \sum_{i=1}^{n} \text{Var}(X_i) \), by verifying the Lindeberg condition or the Lyapunov condition.

The Beta(\( \alpha, \beta \)) distribution has expectation \( \alpha / (\alpha + \beta) \), variance \( \alpha \beta / [(1 + \alpha + \beta)(\alpha + \beta)^2] \), and density \( \Gamma(\alpha + \beta)x^{\alpha - 1}(1-x)^{\beta - 1}I \{ 0 < x < 1 \} / \Gamma(\alpha)\Gamma(\beta) \).

### Problem 10
Let \( X_1, \ldots, X_n \) be an iid sample from Poisson(\( \theta \)). Throughout this problem, you may assume that the Poisson distribution satisfies all relevant regularity conditions.

(a) Show that the Jeffreys prior on \( (0, \infty) \) is the improper prior density \( \lambda(\theta) = 1/\sqrt{\theta} \). To do this, it suffices to show that \( \lambda(\theta) \) is proportional to \( \sqrt{I(\theta)} \).

The Poisson(\( \theta \)) distribution has expectation \( \theta \), variance \( \theta \), and mass function \( p(x) = e^{-\theta} \theta^x / x! \) for \( x \) a nonnegative integer.

(b) Show that with the improper Jeffreys prior \( \lambda(\theta) = 1/\sqrt{\theta} \), the posterior distribution of \( \theta \) is gamma. Find the Bayes estimator \( \delta_n = \text{E}(\theta | X_1, \ldots, X_n) \) using this prior. Finally, give the asymptotic distribution of \( \sqrt{n}(\delta_n - \theta) \).

The Gamma(\( \alpha, \beta \)) distribution has expectation \( \alpha / \beta \), variance \( \alpha / \beta^2 \), and density \( \beta^\alpha x^{\alpha - 1}e^{-\beta x}I \{ x > 0 \} / \Gamma(\alpha) \).