Suppose that we wish to test the hypotheses

\[ H_0 : \theta = \theta_0 \quad \text{and} \quad H_1 : \theta > \theta_0. \] (61)

The test is to be based on a statistic \( T_n \), where as always \( n \) denotes the sample size, and we shall decide to reject \( H_0 \) in (61) if \( T_n \geq C_n \) (62) for some constant \( C_n \). We may define some basic asymptotic concepts regarding tests of this type.

**Definition 23.1** If \( P_{\theta_0}(T_n \geq C_n) \to \alpha \) for test (62), then test (62) is said to have asymptotic level \( \alpha \).

**Definition 23.2** If two different tests of the same hypotheses reach the same conclusion with probability approaching 1 under the null hypothesis as \( n \to \infty \), the tests are said to be asymptotically equivalent.

As usual, the power of test (62) under the alternative \( \theta \) is defined to be

\[ \beta_n(\theta) = P_{\theta}(T_n \geq C_n). \]

Naturally, we expect that the power should approach 1.

**Definition 23.3** A test (or, more precisely, a sequence of tests) is consistent against the alternative \( \theta \) if \( \beta_n(\theta) \to 1 \).

Unfortunately, the concepts we have defined so far are of limited usefulness. If we wish to compare two different tests of the same hypotheses, then if the tests are both sensible they should be asymptotically equivalent and consistent. Thus, consistency is nice but it doesn’t tell us much; asymptotic equivalence is nice but it doesn’t allow us to compare tests.

We make things more interesting by considering, instead of a fixed alternative \( \theta \), a sequence of alternatives \( \theta_1, \theta_2, \ldots \). Suppose for test (62) that for each \( n \), the distribution of \( T_n \) is determined by \( \theta_n \) (i.e., \( \theta_n \) is the “truth”) and that

\[ \frac{\sqrt{n}(T_n - \mu(\theta_n))}{\tau(\theta_n)} \xrightarrow{L} N(0, 1). \] (63)

Furthermore, suppose that under \( H_0 : \theta = \theta_0 \),

\[ \frac{\sqrt{n}(T_n - \mu(\theta_0))}{\tau(\theta_0)} \xrightarrow{L} N(0, 1). \] (64)

Note that \( \mu(\theta) \) is the mean of \( T_n \) assuming that \( \theta \) is the truth. We may calculate the power of the test against the sequence of alternatives \( \{\theta_n\} \) in a straightforward way.

First, we determine a value for \( C_n \) so that test (62) has asymptotic level \( \alpha \). Define \( u_{\alpha} \) to be the constant such that \( 1 - \Phi(u_{\alpha}) = \alpha \), where \( \Phi(t) \) denotes the cdf of a standard normal random variable. By limit (64),

\[ P_{\theta_0} \left\{ T_n - \mu(\theta_0) \geq \frac{\tau(\theta_0)u_{\alpha}}{\sqrt{n}} \right\} \to \alpha; \]
Therefore, we define a new test, namely
\[ \text{reject } H_0 \text{ in } (61) \text{ if } T_n \geq \mu(\theta_0) + \frac{\tau(\theta_0)\nu_n}{\sqrt{n}} \]  
and conclude that test (65) has asymptotic level \( \alpha \) as desired.

We now calculate the power of test (65) against the alternative \( \theta_n \):
\[
\beta_n(\theta_n) = P_{\theta_n} \left\{ T_n \geq \mu(\theta_0) + \frac{\tau(\theta_0)\nu_n}{\sqrt{n}} \right\} = P_{\theta_n} \left\{ \sqrt{n}(\frac{T_n - \mu(\theta_n)}{\tau(\theta_n)}) + \sqrt{n}(\theta_n - \theta_0) \cdot \frac{\mu(\theta_n) - \mu(\theta_0)}{\theta_n - \theta_0} \geq u_\alpha \right\}. 
\]

Thus, \( \beta_n(\theta_n) \) tends to an interesting limit (i.e., a limit between \( \alpha \) and 1) if \( \tau(\theta_n) \to \tau(\theta_0); \sqrt{n}(\theta_n - \theta_0) \) tends to a nonzero, finite limit; and \( \mu(\theta) \) is differentiable at \( \theta_0 \). This fact is summarized in the following theorem, which is similar to Theorem 3.3.3 on page 159 of Lehmann.

**Theorem 23.1** Let \( \theta_n > \theta_0 \) for all \( n \). Suppose that limits (63) and (64) hold, \( \tau(\theta) \) is continuous at \( \theta_0 \), \( \mu(\theta) \) is differentiable at \( \theta_0 \), and \( \sqrt{n}(\theta_n - \theta_0) \to \Delta \) for some finite \( \Delta > 0 \). If \( \mu'(\theta_0) \) or \( \tau(\theta_0) \) depend on \( n \), then suppose that \( \mu'(\theta_0)/\tau(\theta_0) \) tends to a nonzero, finite limit. Then if \( \beta_n(\theta_n) \) denotes the power of test (65) against the alternative \( \theta_n \),
\[
\beta_n(\theta_n) \to \lim_{n \to \infty} \Phi \left( \frac{\Delta \mu'(\theta_0)}{\tau(\theta_0)} - u_\alpha \right).
\]

The proof of Theorem 23.1 merely uses Equation (66) and Slutsky’s theorem, since the hypotheses of the theorem imply that \( \tau(\theta_n)/\tau(\theta_0) \to 1 \) and \( \{\mu(\theta_n) - \mu(\theta_0)\}/(\theta_n - \theta_0) \to \mu'(\theta_0) \).

**Example 23.1** Let \( X \sim \text{Binomial}(n, p_n) \), where \( p_n = p_0 + \Delta/\sqrt{n} \) and \( T_n = X/n \). To test \( H_0 : p = p_0 \) against \( H_1 : p > p_0 \), note that
\[
\frac{\sqrt{n}(T_n - p_0)}{\sqrt{p_0(1 - p_0)}} \overset{\text{d}}{\to} N(0, 1)
\]
under \( H_0 \). Thus, test (65) says to reject \( H_0 \) whenever \( T_n \geq p_0 + u_\alpha \sqrt{p_0(1 - p_0)}/n \). This test has asymptotic level \( \alpha \). Since clearly \( \tau(p) = \sqrt{p(1 - p)} \) is continuous and \( \mu(p) = p \) is differentiable, Theorem 23.1 applies in this case as long as we can verify the limit (63).

Let \( X_1, \ldots, X_n \) be iid Bernoulli(\( p_n \)). Then if \( X_1 - p_n, \ldots, X_n - p_n \) can be shown to satisfy the Lyapunov condition, we have
\[
\frac{\sqrt{n}(T_n - p_n)}{\tau(p_n)} \overset{\text{d}}{\to} N(0, 1)
\]
and so Theorem 23.1 applies. The Lyapunov condition is easy to verify, since \( |X_1 - p_n| \leq 1 \) implies
\[
\frac{1}{\text{Var}(nT_n)^{3/2}} \sum_{i=1}^{n} E |X_{ni} - p_n|^3 \leq \frac{n}{\{np_n(1 - p_n)\}^{3/2}} \to 0 .
\]

Thus, we conclude by Theorem 23.1 that
\[
\beta_n(p_n) \to \Phi \left( \frac{\Delta}{\sqrt{p_0(1 - p_0)}} - u_{\alpha} \right).
\]
To apply this result, suppose that we wish to test whether a coin is fair by flipping it 100 times. We reject \( H_0: p = 1/2 \) in favor of \( H_1: p > 1/2 \) if the number of successes divided by 100 is at least as large as \( 1/2 + u_{0.05}/20 \), or 0.582. The power of this test against the alternative \( p = 0.6 \) is approximately

\[
\Phi \left( \frac{\sqrt{100(0.6 - 0.5)}}{\sqrt{0.5^2}} - 1.645 \right) = \Phi(2 - 1.645) = 0.639.
\]

Compare this asymptotic approximation with the exact power, which in this case is easy to compute: The probability of at least 59 successes out of 100 for a binomial(100, 0.6) random variable is 0.623.

As seen in Example 23.1, the asymptotic result of Theorem 23.1 may be applied to a fixed sample size \( n \) and a fixed alternative \( \theta \) as follows:

\[
\beta_n(\theta) \approx \Phi \left( \frac{\sqrt{n} (\mu(\theta) - \mu(\theta_0))}{\tau(\theta_0)} - u_\alpha \right) \tag{67}
\]

There is an alternative formulation that yields a slightly different approximation. Starting from

\[
\beta_n(\theta) = P_0 \left\{ \frac{\sqrt{n} (T_n - \mu(\theta))}{\tau(\theta)} \geq u_\alpha \frac{\tau(\theta_0)}{\tau(\theta)} - \frac{\sqrt{n} (\mu(\theta) - \mu(\theta_0))}{\tau(\theta)} \right\},
\]

we obtain

\[
\beta_n(\theta) \approx \Phi \left( \frac{\sqrt{n} (\mu(\theta) - \mu(\theta_0))}{\tau(\theta)} - u_\alpha \frac{\tau(\theta_0)}{\tau(\theta)} \right). \tag{68}
\]

Applying approximation (68) to the binomial case of Example 23.1, we obtain the value 0.641 for the approximate power.

We may invert approximations (67) and (68) to obtain approximate sample sizes required to achieve desired power \( \beta \) against alternative \( \theta \). From (67) we obtain

\[
\sqrt{n} \approx \frac{(u_\alpha - u_\beta) \tau(\theta_0)}{\mu(\theta) - \mu(\theta_0)} \tag{69}
\]

and from (68) we obtain

\[
\sqrt{n} \approx \frac{u_\alpha \tau(\theta_0) - u_\beta \tau(\theta)}{\mu(\theta) - \mu(\theta_0)}. \tag{70}
\]

Problems

**Problem 23.1** Let \( P_\theta \) be a family of probability distributions indexed by a real parameter \( \theta \). If \( X \sim P_\theta \), define \( \mu(\theta) = \mathbb{E}(X) \) and \( \sigma^2(\theta) = \text{Var}(X) \). Now let \( \theta_1, \theta_2, \ldots \) be a sequence of parameter values such that \( \theta_n \to \theta_0 \) as \( n \to \infty \). Suppose that for each \( n \), \( X_{n1}, \ldots, X_{nn} \) are iid from \( P_{\theta_n} \) and define \( \overline{X}_n = \sum_{i=1}^{n} X_{ni}/n \). Prove that if \( \sigma^2(\theta_n) < \infty \) and \( \sigma^2(\theta) \) is continuous at the point \( \theta_0 \), then

\[
\sqrt{n} [\overline{X}_n - \mu(\theta_n)] \overset{\mathcal{L}}{\to} N(0, \sigma^2(\theta_0))
\]

as \( n \to \infty \).

**Hint:** Verify the Lindeberg condition.
Problem 23.2  Suppose $X_1, X_2, \ldots$ are iid exponential random variables with mean $\theta$. Consider the test of $H_0 : \theta = 1$ vs $H_1 : \theta > 1$ in which we reject $H_0$ when

$$\bar{X}_n \geq 1 + \frac{u_\alpha}{\sqrt{n}},$$

where $\alpha = .05$.

(a) Derive an asymptotic approximation to the power of the test. Tell where you use the result of Problem 23.1.

(b) Because the sum of iid exponential random variables is a gamma random variable, it is possible to compute the power exactly in this case. Create a table in which you compare the exact power of the test against the alternative $\theta = 1.2$ to the asymptotic approximation in part (a) for $n \in \{5, 10, 15, 20\}$.

Problem 23.3  Let $X_1, \ldots, X_n$ be iid from Poisson ($\lambda$). The asymptotic power given in Expression (3.3.19) on p. 163 is derived from equation (3.3.11). Derive a different approximation using equation (3.3.12). Then create a table in which you list the exact power and each of the two asymptotic approximations for the test that rejects $H_0 : \lambda = 1$ in favor of $H_1 : \lambda > 1$ when

$$\frac{\sqrt{n}(\bar{X}_n - \lambda_0)}{\sqrt{\lambda_0}} \geq u_\alpha,$$

where $n = 20$ and $\alpha = .05$, against each of the alternatives 1.1, 1.5, and 2.

Problem 23.4  Let $X_1, \ldots, X_n$ be an independent sample from an exponential distribution with mean $\lambda$, and $Y_1, \ldots, Y_n$ be an independent sample from an exponential distribution with mean $\mu$. Assume that $X_i$ and $Y_i$ are independent. We are interested in testing the hypothesis $H_0 : \lambda = \mu$ versus $H_1 : \lambda > \mu$. Consider the statistic

$$T_n = 2 \sum_{i=1}^{n} (I_i - 1/2)/\sqrt{n},$$

where $I_i$ is the indicator variable $I_i = I(X_i > Y_i)$.

(a) Derive the asymptotic distribution of $T_n$ under the null hypothesis.

(b) Use the Lindeberg Theorem to show that, under the local alternative hypothesis $(\lambda_n, \mu_n) = (\lambda + n^{-1/2} \delta, \lambda)$, where $\delta > 0$,

$$\frac{\sum_{i=1}^{n}(I_i - \rho_n)}{\sqrt{n\rho_n(1 - \rho_n)}} \xrightarrow{\text{d}} N(0, 1),$$

where $\rho_n = \frac{\lambda_n}{\lambda_n + \mu_n} = \frac{\lambda + n^{-1/2} \delta}{2\lambda + n^{-1/2} \delta}$.

(c) Using the conclusion of part (b), derive the asymptotic distribution of $T_n$ under the local alternative specified in (b).

Problem 23.5  Suppose $X_1, \ldots, X_m$ is an iid sample and $Y_1, \ldots, Y_n$ is an iid sample independent of the $X_i$, with $P(X_i \leq t) = t^2$ for $t \in [0, 1]$ and $P(Y_i \leq t) = (t - \theta)^2$ for $t \in [\theta, \theta + 1]$. Assume $m/(m + n) \to \rho$ as $m, n \to \infty$ and $0 < \theta < 1$.

(a) Solve Problem 3.2.2 on page 206.

(b) Find the asymptotic distribution of $\sqrt{m + n}[g(Y_i - \bar{X}) - g(\theta)]$. 

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24. Asymptotic relative efficiency

We may compare tests by considering the relative sample sizes necessary to achieve the same power at the same level against the same alternative.

**Definition 24.1** Given tests 1 and 2 of the same hypotheses with asymptotic level $\alpha$ and a sequence of alternatives $\{\theta_k\}$, suppose that

$$\beta^{(1)}_{m_k}(\theta_k) \rightarrow \beta$$

and

$$\beta^{(2)}_{n_k}(\theta_k) \rightarrow \beta$$

for sequences $\{m_k\}$ and $\{n_k\}$ of sample sizes. Then the asymptotic relative efficiency (ARE) of test 1 with respect to test 2 is

$$e_{1,2} = \lim_{k \to \infty} \frac{n_k}{m_k},$$

assuming this limit exists.

In Examples 24.1 and 24.2, we consider two different tests for the same hypotheses. Then, in Example 24.3, we compute their asymptotic relative efficiency.

**Example 24.1** Suppose we have paired data $(X_1, Y_1), \ldots, (X_n, Y_n)$. Let $Z_i = Y_i - X_i$ for all $i$. Assume that the $Z_i$ are iid with cdf $P(Z_i \leq z) = F(z - \theta)$ for some $\theta$, where $f(z) = F'(z)$ exists and is symmetric about 0. Let $W_1, \ldots, W_n$ be a permutation of $Z_1, \ldots, Z_n$ such that $|W_1| \leq |W_2| \leq \cdots \leq |W_n|$.

We wish to test $H_0 : \theta = 0$ against $H_1 : \theta > 0$. First, consider the Wilcoxon signed rank test. Define

$$R_n = \sum_{i=1}^{n} i I\{W_i > 0\}.$$ 

Then under $H_0$, the random variables $I\{W_i > 0\}$ are iid Bernoulli(1/2). Thus,

$$E R_n = \sum_{i=1}^{n} \frac{i}{2} = \frac{n(n+1)}{4}$$

and

$$\text{Var} R_n = \sum_{i=1}^{n} \frac{i^2}{4} = \frac{n(n+1)(2n+1)}{24}.$$ 

Furthermore, it is easy to show (by verifying the Lyapunov condition) that

$$\frac{R_n - E R_n}{\sqrt{\text{Var} R_n}} \xrightarrow{\mathcal{D}} N(0, 1)$$

under $H_0$. Thus, a test with asymptotic level $\alpha$ rejects $H_0$ when

$$R_n \geq \frac{n(n+1)}{4} + \frac{u_{\alpha} \tau(0)}{\sqrt{n}},$$

where $\tau(0) = n \sqrt{(n+1)(2n+1)/24}$. Now we must find $E R_n$ under the alternative $\theta_n = \Delta/\sqrt{n}$.

First, we note that since $|W_i| \leq |W_j|$ for $i \leq j$, $W_i + W_j > 0$ if and only if $W_j > 0$. Therefore, $\sum_{i=1}^{j} I\{W_i + W_j = 0\} = j I\{W_j > 0\}$ and so we may rewrite $R_n$ in the form

$$R_n = \sum_{j=1}^{n} \sum_{i=1}^{j} I\{W_i + W_j > 0\} = \sum_{j=1}^{n} \sum_{i=1}^{j} I\{Z_i + Z_j > 0\}. \quad (71)$$
The reason equation (71) is true is that for \( i \leq j \), \(|Z_i| \leq |Z_j|\) and so \( Z_i + Z_j > 0 \) if and only if \( Z_j > 0 \). Therefore, from equation (71), we obtain
\[
\mu(\theta_n) = \sum_{j=1}^{n} \sum_{i=1}^{j} P_{\theta_n}(Z_i + Z_j > 0) = n P_{\theta_n}(Z_1 > 0) + \left(\frac{n}{2}\right) P_{\theta_n}(Z_1 + Z_2 > 0).
\]
Since \( P_{\theta_n}(Z_1 > 0) = P_{\theta_n}(Z_1 - \theta_n > -\theta_n) = 1 - F(-\theta_n) \) and
\[
P_{\theta_n}(Z_1 + Z_2 > 0) = P_{\theta_n}\{ (Z_1 - \theta_n) + (Z_2 - \theta_n) > -2\theta_n \}
= E P_{\theta_n}\{ Z_1 - \theta_n > -2\theta_n - (Z_2 - \theta_n) | Z_2 \}
= \int_{-\infty}^{\infty} \{1 - F(-2\theta_n - z)\} f(z) \, dz,
\]
we conclude that
\[
\mu'(\theta) = n f(\theta) + \left(\frac{n}{2}\right) \int_{-\infty}^{\infty} 2 f(-2\theta - z)f(z) \, dz.
\]
Thus, because \( f(-z) = f(z) \) by assumption,
\[
\mu'(0) = nf(0) + n(n-1) \int_{-\infty}^{\infty} f^2(z) \, dz.
\]
Letting
\[
K = \int_{-\infty}^{\infty} f^2(z) \, dz,
\]
we obtain
\[
\lim_{n \to \infty} \frac{\mu'(0)}{\tau(0)} = \lim_{n \to \infty} \frac{\sqrt{24\{f(0) + (n-1)K\}}}{\sqrt{(n+1)(2n+1)}} = K \sqrt{12}.
\]
Theorem 23.1 shows that
\[
\beta_n(\theta_n) \to \Phi(\Delta K \sqrt{12} - u_\alpha).
\] (72)
Note that it is necessary to check limit (63) with \( R_n \) in place of \( T_n \), which may be accomplished by verifying that the Lindeberg condition is satisfied.

**Example 24.2** As in Example 24.1, suppose we have paired data \((X_1, Y_1), \ldots, (X_n, Y_n)\) and \(Z_i = Y_i - X_i\) for all \(i\). The \(Z_i\) are iid with cdf \(P(Z_i \leq z) = F(z - \theta)\) for some \(\theta\), where \(f(z) = F'(z)\) exists and is symmetric about 0. Suppose that the variance of \(Z_i\) is \(\sigma^2\).

Since the t-test (unknown variance) and z-test (known variance) have the same asymptotic properties, let’s consider the z-test for simplicity. Then \(\tau(\theta) = \sigma\) for all \(\theta\). The relevant statistic is merely \(\bar{Z}_n\), and the central limit theorem implies that \(\sqrt{n}Z_n / \sigma \overset{L}{\to} N(0,1)\) under the null hypothesis. Therefore, the z-test in this case rejects \(H_0: \theta = 0\) in favor of \(H_1: \theta > 0\) whenever \(\bar{Z}_n > u_\alpha \sigma / \sqrt{n}\). It is easy (by verifying the Lindeberg condition) to show that
\[
\frac{\sqrt{n}(\bar{Z}_n - \theta_n)}{\sigma} \overset{L}{\to} N(0,1)
\]
under the alternatives \(\theta_n = \Delta / \sqrt{n}\). Therefore, by Theorem 23.1, we obtain
\[
\beta_n(\theta_n) \to \Phi(\Delta / \sigma - u_\alpha)
\] (73)
since \(\mu'(\theta) = 1\) for all \(\theta\).
Before finding the asymptotic relative efficiency (ARE) of the Wilcoxon signed rank test and the t-test, we prove a lemma that enables this calculation very easily.

Suppose that for two tests, called test 1 and test 2, we use sample sizes $m$ and $n$, respectively. We want $m$ and $n$ to tend to infinity together, an idea we make explicit by setting $m = m_k$ and $n = n_k$ for $k = 1, 2, \ldots$. Suppose that we wish to apply both tests to the same sequence of alternative hypotheses $\theta_1, \theta_2, \ldots$. As usual, we make the assumption that $(\theta_k - \theta_0)$ times the square root of the sample size tends to a finite, nonzero limit as $k \to \infty$. Thus, we assume

$$\sqrt{m_k}(\theta_k - \theta_0) \to \Delta_1 \quad \text{and} \quad \sqrt{n_k}(\theta_k - \theta_0) \to \Delta_2.$$  

Then if Theorem 23.1 may be applied to both tests, define $c_1 = \lim \mu_1'(\theta_0)/\tau_1(\theta_0)$ and $c_2 = \lim \mu_2'(\theta_0)/\tau_2(\theta_0)$. The theorem says that

$$\beta_{m_k}(\theta_k) \to \lim_{k \to \infty} \Phi \left\{ \sqrt{m_k}(\theta_k - \theta_0)c_1 - u_\alpha \right\} \quad \text{and} \quad \beta_{n_k}(\theta_k) \to \lim_{k \to \infty} \Phi \left\{ \sqrt{n_k}(\theta_k - \theta_0)c_2 - u_\alpha \right\}. \quad (74)$$

To find the ARE, then, Definition 24.1 specifies that we assume that the two limits in (74) are the same, which forces

$$\frac{n_k}{m_k} \to \frac{c_1^2}{c_2^2}.$$  

Thus, the ARE of test 1 with respect to test 2 equals $(c_1/c_2)^2$. This result is summed up in the following lemma, which defines a new term, efficacy.

**Lemma 24.1** For a test to which Theorem 23.1 applies, define the efficacy of the test to be

$$c = \lim \mu'(\theta_0)/\tau(\theta_0). \quad (75)$$

Suppose that Theorem 23.1 applies to each of two tests, called test 1 and test 2. Then the ARE of test 1 with respect to test 2 equals $(c_1/c_2)^2$.

**Example 24.3** Using the results of Examples 24.1 and 24.2, we conclude that the efficacies of the Wilcoxon signed rank test and the t-test are

$$\sqrt{12} \int_{-\infty}^{\infty} f^2(z) \, dz \quad \text{and} \quad \frac{1}{\sigma},$$

respectively. Thus, Lemma 24.1 implies that the ARE of the signed rank test to the t-test equals

$$12\sigma^2 \left( \int_{-\infty}^{\infty} f^2(z) \, dz \right)^2.$$  

In the case of normally distributed data, we may verify without too much difficulty that the integral above equals $2(2\sigma^2 \sqrt{\pi})^{-1}$, so the ARE is $3/\pi \approx 0.9549$. Notice how close this is to one, suggesting that for normal data, we lose very little efficiency by using a signed rank test instead of a t-test. In fact, Lehmann states (without proof) in the textbook that this asymptotic relative efficiency has a lower bound of 0.864. However, there is no upper bound on the ARE in this case, which means that examples exist for which the t-test is arbitrarily inefficient compared to the signed rank test.

**Problems**
Problem 24.1  For the hypotheses considered in Examples 24.1 and 24.2, the sign test is based on the statistic \( N_+ = \# \{ i : Z_i > 0 \} \). Since \( 2\sqrt{n}(N_+/n - 1/2) \xrightarrow{d} N(0,1) \) under the null hypothesis, the sign test (with continuity correction) rejects \( H_0 \) when

\[
N_+ - \frac{1}{2} \geq \frac{\alpha \sqrt{n}}{2} + \frac{n}{2}.
\]

(a) Find the efficacy of the sign test. Make sure to indicate how you go about verifying equation (63).

(b) Find the ARE of the sign test with respect to the signed rank test and the t-test. Evaluate each of these for the case of normal data.

Problem 24.2  (a) Do Problem 4.4 on p. 211. Take \( a = 0 \) and \( b = 1 \) in each case.

(b) Do Problem 4.6 on p. 211 (give the limiting values of the three ARE’s as a function of \( \epsilon \)). What does this result tell you about the 1-sample t-test?

Problem 24.3  Suppose \( X_1, \ldots, X_n \) are iid from a uniform \((0, 2\theta)\) distribution. We wish to test \( H_0 : \theta = \theta_0 \) against \( H_1 : \theta > \theta_0 \) at \( \alpha = .05 \).

(a) Define \( Q_1 \) and \( Q_3 \) to be the first and third quartiles of the sample. Consider test A, which rejects when

\[
Q_3 - Q_1 - \theta_0 \geq A_n,
\]

and test B, which rejects when

\[
\bar{X} - \theta_0 \geq B_n.
\]

Based on the asymptotic distribution of \( \bar{X} \) and the joint asymptotic distribution of \((Q_1, Q_3)\), find the values of \( A_n \) and \( B_n \) that correspond with the test in (65). Then find the asymptotic relative efficiency of test A relative to test B.

(b) Solve Problem 4.20(ii) on p. 213. Use the same values of \( \alpha \) and \( \beta \) as in Table 3.4.2 on p. 185 and give answers accurate to 2 decimal places.