Question 1  Since $X$ is a binomial($3, \frac{2}{3}$) random variable, its marginal pmf is given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{27}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{2}{9}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{9}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{8}{27}$</td>
</tr>
</tbody>
</table>

Furthermore, for any $y \leq x$,

$$P(Y = y|X = x) = \binom{x}{y} (0.6)^y (0.4)^{x-y}.$$  

(a) $P(X = 3, Y = 2) = P(Y = 2|X = 3)P(X = 3) = \binom{3}{2} (0.6)^2 (0.4)^1 \left(\frac{8}{27}\right) = 0.128$

(b) Whenever $y \leq x$, we can find $P(X = x, Y = y)$ as in part (a). Whenever $y > x$, clearly $P(X = x, Y = y) = 0$. Thus, we obtain the following joint pmf for $X$ and $Y$:

<table>
<thead>
<tr>
<th>$p(x, y)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0</td>
<td>0.037</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.089</td>
<td>0.133</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.071</td>
<td>0.213</td>
<td>0.160</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.019</td>
<td>0.085</td>
<td>0.128</td>
<td>0.064</td>
</tr>
</tbody>
</table>

(c) The marginal pmf for $X$ is given above. The marginal pmf for $Y$ is obtained by summing the columns of the table in part (b):

<table>
<thead>
<tr>
<th>$y$</th>
<th>$P(Y = y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.216</td>
</tr>
<tr>
<td>1</td>
<td>0.432</td>
</tr>
<tr>
<td>2</td>
<td>0.288</td>
</tr>
<tr>
<td>3</td>
<td>0.064</td>
</tr>
</tbody>
</table>

(By the way, notice that $Y$ is a binomial($3, 0.4$) random variable. Do you see why? In general, if $X$ is binomial($n, p$) and $Y|X$ is binomial($X, q$), then $Y$ is binomial($n, r$). What is $r$?)

Question 2 (a)

$$f_X(x) = \int_0^\infty xe^{-x} e^{-xy} dy = xe^{-x} \left(\frac{1}{-xe^{-xy}}\right)\bigg|_0^\infty = e^{-x}.$$  

Therefore,

$$P(X > 2) = \int_2^\infty e^{-x} dx = -e^{-x}\bigg|_2^\infty = e^{-2} = 0.135.$$

(b) Using the change of variables $z = x(1 + y)$ gives

$$f_Y(y) = \int_0^\infty xe^{-x(1+y)} dx = \frac{1}{(1+y)^2} \int_0^\infty ze^{-z} dz = \frac{1}{(1+y)^2} \Gamma(2) = \frac{1}{(1+y)^2}.$$  

Therefore,

$$P(Y > 2) = \int_2^\infty \frac{dy}{(1+y)^2} = \left.-\frac{1}{1+y}\right|_2^\infty = \frac{1}{3}.$$  

(c) The probability that both $X$ and $Y$ exceed 2 is

$$P(X > 2 \text{ and } Y > 2) = \int_2^\infty xe^{-x} \int_2^\infty e^{-xy} dy dx = \int_2^\infty e^{-x} e^{-2x} dx$$

$$= -\left.\frac{1}{3}e^{-3x}\right|_2^\infty = \frac{e^{-6}}{3} = 0.000826.$$
Therefore, that the probability that at least one exceeds 2 is
\[ P(X > 2 \text{ or } Y > 2) = P(X > 2) + P(Y > 2) - P(X > 2 \text{ and } Y > 2) \]
\[ = e^{-2} + \frac{1}{3} - \frac{e^{-6}}{3} = 0.468. \]

**Question 3 (a)** Since \( X \) is binomial \((3, \frac{2}{3})\), \( E(X) = 3 \times \frac{2}{3} = 2 \). From part (c) of the solution to Question 1, we can find \( E(Y) = 1.2 \).

There are 7 possible values of \( XY \), namely 0, 1, 2, 3, 4, 6, and 9. Since \( XY = 0 \) in this problem if and only if \( Y = 0 \), \( P(XY = 0) = P(Y = 0) = 0.216 \). The probabilities of the other six possibilities come directly from the joint mass function table in the solution to Question 1:

<table>
<thead>
<tr>
<th>( xy )</th>
<th>( P(XY = xy) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.216</td>
</tr>
<tr>
<td>1</td>
<td>0.133</td>
</tr>
<tr>
<td>2</td>
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<td>6</td>
<td>0.128</td>
</tr>
<tr>
<td>9</td>
<td>0.064</td>
</tr>
</tbody>
</table>

Thus, \( E(XY) = 0(0.216) + 1(0.133) + 2(0.213) + 3(0.085) + 4(0.160) + 6(0.128) + 9(0.064) = 2.8 \).

This means that \( \text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 2.8 - 2(1.2) = 2.8 - 2.4 = 0.4 \).

**Question 4** Since we already know the marginal density \( f_X(x) \) of \( X \) from Question 2, we can find \( E(X) \) by integrating \( x f_X(x) \) dx:

\[ \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{0}^{\infty} x e^{-x} \, dx = \Gamma(2) = 1. \]

Alternatively, we could have found \( E(X) = 1 \) by evaluating \( \iint \! \! \! \! \! \iint x f(x, y) \, dy \, dx \).

There is only one way to find \( E(XY) \): We must evaluate

\[ E(XY) = \int_{0}^{\infty} \int_{0}^{\infty} xy f(x, y) \, dy \, dx = \int_{0}^{\infty} x^2 e^{-x} \int_{0}^{\infty} e^{-xy} \, dy \, dx \]
\[ = \int_{0}^{\infty} x^2 e^{-x} \left( \frac{1}{x} \right) \, dx = \int_{0}^{\infty} xe^{-x} \, dx = \Gamma(2) = 1. \]

Note that \( E(Y) \) is undefined here, and hence \( \text{Cov}(X, Y) \) is undefined, because

\[ \int_{0}^{\infty} \frac{y}{(1+y)^2} \, dy \]

is infinite.

**Question 5 (a)** Since the total volume shipped equals \( 27X_1 + 125X_2 + 512X_3 \), this question is asking for \( E(27X_1 + 125X_2 + 512X_3) \) and \( V(27X_1 + 125X_2 + 512X_3) \):

\[
E(27X_1 + 125X_2 + 512X_3) = 27E(X_1) + 125E(X_2) + 512E(X_3) = 27 \times 200 + 125 \times 250 + 512 \times 100 = 87,850.
\]

The units of this expectation are \( \text{ft}^3 \). By independence, we can say that

\[
V(27X_1 + 125X_2 + 512X_3) = 27^2 V(X_1) + 125^2 V(X_2) + V(X_3) = 729 \times 100 + 15625 \times 144 + 262144 \times 64 = 19,100,116.
\]
The units of this variance are \((\text{ft}^3)^2\).

(b) Independence does not enter into the calculation of the expectation, but the calculation of the variance relies upon the assumption of independence. Thus, if the \(X_i\)'s were not independent, the expectation calculation would still be correct but the variance calculation probably would not.

**Question 6 (a)** The first thing to notice is that \(\overline{X}\) and \(\overline{Y}\) are linear combinations of normal random variables. Therefore, they are themselves normal random variables. This means that their distributions are completely specified if we find their means and variances.

\[
E(\overline{X}) = E\left(\frac{1}{20}X_1 + \cdots + \frac{1}{20}X_{20}\right)
\]
\[
= \frac{1}{20} E(X_1) + \cdots + \frac{1}{20} E(X_{20}) = \frac{1}{20} (120 + \cdots + 120) = 120.
\]

A similar calculation finds that \(E(\overline{Y}) = 150\). Furthermore, since \(V(X_i) = 100\), the independence of \(X_1, \ldots, X_{20}\) implies

\[
V(\overline{X}) = V\left(\frac{1}{20}X_1 + \cdots + \frac{1}{20}X_{20}\right)
\]
\[
= \left(\frac{1}{20}\right)^2 V(X_1) + \cdots + \left(\frac{1}{20}\right)^2 V(X_{20}) = \frac{1}{20^2} (20 \times 100) = \frac{100}{20} = 5.
\]

A similar calculation reveals that \(V(\overline{Y}) = \frac{425}{10} = 22.5\).

In conclusion, we have found that \(\overline{X}\) is normally distributed with mean 120 and variance 5, and \(\overline{Y}\) is normally distributed with mean 150 and variance 22.5.

(b) As in part (a), since \(\overline{X} - \overline{Y}\) is a linear combination of normal random variables, it is also a normal random variable. Furthermore, \(E(\overline{X} - \overline{Y}) = E(\overline{X}) - E(\overline{Y}) = 120 - 150 = -30\). Because the \(X\)'s are independent of the \(Y\)'s, we can claim that \(V(\overline{X} - \overline{Y}) = V(\overline{X}) + V(\overline{Y})\) (note: not \(V(\overline{X}) - V(\overline{Y})\)). Therefore, \(\overline{X} - \overline{Y}\) is a normal random variable with mean \(-30\) and variance 27.5.

(c) From part (b), we know that \(\overline{X} - \overline{Y}\) is a normal random variable with mean \(-30\) and standard deviation \(\sqrt{27.5} = 5.24\). Thus,

\[
P(-40 \leq \overline{X} - \overline{Y} \leq -10) = P\left(\frac{-40 + 30}{5.24} \leq \frac{\overline{X} - \overline{Y} + 30}{5.24} \leq \frac{-10 + 30}{5.24}\right)
\]
\[
= P(-1.91 \leq Z \leq 3.81) = \Phi(3.81) - \Phi(-1.91).
\]

Note that 3.81 is not in the table in the book. Thus, we can approximate \(\Phi(3.81) \approx 1\), which gives an answer of \(1 - .0281 = .9719\).