STATISTICS 401, Section 001
Homework for Chapter 3
SOLUTIONS

Question 1 (a) Since \( p(y) = ky^2 \) and \( \sum_{i=1}^{5} p(y) = 1 \), we obtain

\[
1k + 4k + 9k + 16k + 25k = 55k = 1,
\]

which implies \( k = \frac{1}{55} \).

(b) “At most three” means 1, 2, or 3 in this case, so we add up the corresponding probabilities and obtain

\[
\frac{1}{55} + \frac{4}{55} + \frac{9}{55} = \frac{14}{55}.
\]

(c) “Between two and four forms (inclusive)” means 2, 3, or 4 in this case, so we add up the corresponding probabilities and obtain

\[
\frac{4}{55} + \frac{9}{55} + \frac{16}{55} = \frac{29}{55}.
\]

Question 2 It’s probably easiest to start by writing out the pmf of \( X \). This is done by noting the sizes of the jumps in the value of \( F(x) \) wherever those jumps occur.

We can see that the jumps occur at 0, 1, 2, 3, 4, 5, 6, and 7. The sizes of the jumps may be obtained by subtraction, which gives as the pmf

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P(X = x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.15</td>
</tr>
<tr>
<td>1</td>
<td>.24</td>
</tr>
<tr>
<td>2</td>
<td>.23</td>
</tr>
<tr>
<td>3</td>
<td>.15</td>
</tr>
<tr>
<td>4</td>
<td>.10</td>
</tr>
<tr>
<td>5</td>
<td>.07</td>
</tr>
<tr>
<td>6</td>
<td>.05</td>
</tr>
<tr>
<td>7</td>
<td>.01</td>
</tr>
</tbody>
</table>

The answers below come directly from the pmf table above:

(a) \( .23 \)

(b) \( .10 + .07 + .05 + .01 = .23 \)

(c) \( .23 + .15 + .10 + .07 = .55 \)

(d) \( .15 + .10 = .25 \)

Question 3 For each fixed value of \( k \), let \( Y_k \) equal the number of magazines actually sold, assuming \( k \) were purchased to begin with. Note that \( Y_k = \min \{ X, k \} \) because we can’t sell more than the demand and we can’t sell more than we purchased to begin with.

(a) Since \$2.70 \times Y_k \$ is the amount of money we take in and \$1.50 \times k \$ is the amount we spend to purchase the magazines, the profit equals \$2.70 \times Y_k \$ − \$1.50 \times k \$.

(b) We need the pmf of \( Y_k \) for each value of \( k \). These are given below. Note that \( Y_k \) is the same as \( X \).

<table>
<thead>
<tr>
<th>( y )</th>
<th>( P(Y_6 = y) )</th>
<th>( P(Y_5 = y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/15</td>
<td>2/15</td>
<td>3/15</td>
</tr>
<tr>
<td>2/15</td>
<td>4/15</td>
<td>5/15</td>
</tr>
</tbody>
</table>

We wish to find the expected value of profit, which equals \( E(2.7Y_k - 1.5k) \). By linearity, this equals \( 2.7E(Y_k) - 1.5k \). Using the pmf’s above, we can easily calculate \( E(Y_6) = 3.80 \), \( E(Y_5) = 3.67 \), \( E(Y_4) = 3.33 \), \( E(Y_3) = 2.73 \), \( E(Y_2) = 1.93 \), and \( E(Y_1) = 1.00 \). Therefore, we obtain

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$2.7 \times 3.80 - 1.5 \times 6 = $1.26 $</td>
</tr>
<tr>
<td>5</td>
<td>$2.7 \times 3.67 - 1.5 \times 5 = $2.40 $</td>
</tr>
<tr>
<td>4</td>
<td>$2.7 \times 3.33 - 1.5 \times 4 = $3.00 $</td>
</tr>
<tr>
<td>3</td>
<td>$2.7 \times 2.73 - 1.5 \times 3 = $2.88 $</td>
</tr>
<tr>
<td>2</td>
<td>$2.7 \times 1.93 - 1.5 \times 2 = $2.22 $</td>
</tr>
<tr>
<td>1</td>
<td>$2.7 \times 1.00 - 1.5 \times 1 = $1.20 $</td>
</tr>
</tbody>
</table>
(c) From the table above, we see that \( k = 4 \) magazines will maximize \( f(k) \).

**Question 4** Since \( Y_6 \) and \( X \) are the same, we know that \( E(X) = 3.8 \) from the previous question. To find the variance, we also need \( E(X^2) \), which equals

\[
1 \times \frac{1}{15} + 4 \times \frac{2}{15} + 9 \times \frac{3}{15} + 16 \times \frac{4}{15} + 25 \times \frac{3}{15} + 36 \times \frac{2}{15} = 16.47.
\]

Therefore, \( V(X) = 16.47 - (3.80)^2 = 2.03 \).

For the second part of the question, we note that if \( 6 \) are originally bought, then the number recycled equals \( 6 - Y_6 \), which is also \( 6 - X \). Thus, we find \( E(6 - X) = 6 - 3.8 = 2.2 \) and \( V(6 - X) = V(X) = 2.03 \).

**Question 5** \( X \) is a binomial(25, .05) random variable.

(a) \[
P(X = 0) = \binom{25}{0} (.05)^0 (.95)^{25} = (.95)^{25} = .277
\]

(b) \[
P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)
\]
\[
= \binom{25}{0} (.05)^0 (.95)^{25} + \binom{25}{1} (.05)^1 (.95)^{24} + \binom{25}{2} (.05)^2 (.95)^{23} + \binom{25}{3} (.05)^3 (.95)^{22}
\]
\[
= .277 + .365 + .231 + .093 = .966
\]

(c) \[
P(X \geq 3) = 1 - P(X = 2) - P(X = 1) - P(X = 0) = 1 - .231 - .365 - .277 = .127
\]

(d) \[
P(1 \leq X < 3) = P(X = 1) + P(X = 2) = .365 + .231 = .596
\]

**Question 6** Let \( X \) denote the number of ten randomly selected customers who want the oversize rackets. Then \( X \sim \text{Bin}(10, .6) \)

(a) \[
P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)
\]
\[
= \binom{10}{6} (.6)^6 (.4)^4 + \binom{10}{7} (.6)^7 (.4)^3 + \binom{10}{8} (.6)^8 (.4)^2 + \binom{10}{9} (.6)^9 (.4)^1 + \binom{10}{10} (.6)^{10} (.4)^0
\]
\[
= .251 + .215 + .121 + .040 + .006 = .633
\]

(b) The standard deviation of \( X \) equals \( \sqrt{10 \times .6 \times .4} = 1.55 \) and the mean of \( X \) equals \( 10 \times .6 = 6 \). Thus, “within one standard deviation of the mean” is the range from 4.45 to 7.55. This range contains the possible values 5, 6, and 7 of \( X \), so the answer is

\[
P(X = 5) + P(X = 6) + P(X = 7) = \binom{10}{5} (.6)^5 (.4)^5 + \binom{10}{6} (.6)^6 (.4)^4 + \binom{10}{7} (.6)^7 (.4)^3
\]
\[
= .201 + .251 + .215 = .666
\]

(c) All of the next ten customers will get what they want as long as \( X \) is between 3 and 7, inclusive. If \( X < 3 \) or \( X > 7 \) then there won’t be enough of one of the types of rackets. Thus, the answer (using part (c)) is

\[
P(X = 3) + P(X = 4) + P(5 \leq X \leq 7) = \binom{10}{3} (.6)^3 (.4)^4 + \binom{10}{4} (.6)^7 (.4)^3 + .666
\]
\[
= .042 + .111 + .666 = .820
\]

**Question 7** \( X \) is a hypergeometric random variable with parameters \((5, 6, 15)\).
(a) The pmf for $X$ is

$$P(X = x) = \binom{6}{x} \binom{9}{5-x} \binom{15}{5}$$

(b) $E(X) = 5 \times \frac{6}{15} = 2$ and $V(X) = \frac{10}{14} \times 5 \times \frac{6}{15} \times \frac{9}{15} = 0.857$.

(c) If $Y \sim \text{binom}(5, 6/15)$, then the expectation of $Y$ is the same as the expectation of $X$ and the variance of $Y$ is $\frac{14}{10}$ times the variance of $X$. Thus, $V(Y) = 1.2$. 