Factorials: Counting rearrangements

Motivating question: Given $n$ objects, how many ways can you order those objects in a list?

Answer: $n!$, read as “$n$ factorial,” which equals $n \times (n-1) \times \cdots \times 2 \times 1$. By the way, we will always define $0!$ to equal 1.

Reasoning: There are $n$ items that can be placed at the top of the list. Then, for each possible choice for the top item, there are $n-1$ items that can be placed in the second position. Then, for each possible choice for the top two items, there are $n-2$ items that can be placed in the third position...

Thus, the total number of possible lists equals $n$ times $n-1$ times $n-2$ times...

Examples:
1. How many ways are there to order the letters in the word LIONS?
2. In a race with 8 participants, how many possible finishes (places 1 through 8) are possible if ties are impossible?

Permutations: Counting subsets when order of selection IS important

Motivating question: Given $n$ objects, how many ways can you order just $k$ of those objects in a list?

Answer: $n!/(n-k)!$, sometimes written as $P^n_k$ or $nP_k$ or $P(n,k)$.

Reasoning: Same as reasoning for $n!$, but we stop after $k$ items on the list. Thus, the total number of possible lists equals $n$ times $n-1$ times $n-2$ times...times $n-k+1$. A short way to write this is $n!/(n-k)!$.

Examples:
1. In an Olympics race with 8 participants, how many possible medal ceremonies (places 1 through 3) are possible if ties are impossible?
2. From a preschool class of 12 students, how many ways can the teacher select a show-and-tell presenter, a snack leader, a fish feeder, and a door opener?

**Combinations: Counting subsets when order of selection IS NOT important**

**Motivating question:** Given \( n \) objects, how many subgroups of size \( k \) can you choose?

**Answer:** \( \binom{n}{k} \), read as “\( n \) choose \( k \),” which equals \( n!/[k!(n-k)!] \). This is sometimes written as \( C_k^n \) or \( nC_k \) or \( C(n,k) \).

**Reasoning:** Once we select the group, if order were important then we could rearrange the group in \( k! \) different ways. Therefore, \( k! \) times \( \binom{n}{k} \) should give \( P_k^n \), or \( n!/(n-k)! \). Divide through by \( k! \) to give the correct number of groups: \( n!/[k!(n-k)!] \).

**Examples:**

1. How many possible 5-card poker hands are possible from a 52-card deck?

2. In the Powerball lottery, each player must select 5 different numbers from 1 through 55. How many ways are there to do this?

3. How many lists of 10 coin flip outcomes result in exactly 4 heads?