p. 783, exercise 12.12 Return to the nematode problem in Exercise 12.10 (which was assigned Nov. 10 and answered in Solution Set 11).

(a) Define the contrast that compares the 0 treatment (the control group) with the average of the other three.

This contrast must put positive weight on the 0 treatment and equal negative weight on the others (or vice versa) in such a way that the weights add to zero. Thus, the contrast is

\[ \Psi = \mu_1 - \frac{1}{3}(\mu_2 + \mu_3 + \mu_4) \]  

(1)

(b) State \( H_0 \) and \( H_a \) for using this contrast to test whether or not the presence of nematodes causes decreased growth in tomato seedlings.

Writing down the null hypothesis for this test (or any contrast test) is easy: \( H_0 : \Psi = 0 \). The alternative, however, requires a bit of thought. Since we’re talking about a one-sided alternative and we’re only interested in the case in which \( \mu_1 \) is larger than the average of \( \mu_2 \), \( \mu_3 \), and \( \mu_4 \), the correct alternative hypothesis is \( H_a : \Psi > 0 \).

(c) Perform the significance test and give the p-value. Do you reject \( H_0 \)?

To answer this question we’ll need the value of the pooled standard deviation \( s_p \). As shown in Solution Set 11, \( s_p = 1.67 \). The significance test will involve the \( t \) statistic \( c/\text{SE}_c \), where \( c \) is the sample value of the contrast and \( \text{SE}_c \) is its standard error. The sample value is easy: Just plug in the sample means in place of the \( \mu \) in (1) to get

\[ c = 10.65 - \frac{1}{3}(10.425 + 5.6 + 5.45) = 3.492. \]

The standard error of \( c \) is given by the formula on page 767. The values of \( a_i \) are 1, \(-1/3\), \(-1/3\), and \(-1/3\). Each value of \( n_i \) is 4. Thus, we get

\[ \text{SE}_c = s_p \sqrt{\sum \frac{a_i^2}{n_i}} = 1.67 \sqrt{\frac{1}{4} + \frac{1/9}{4} + \frac{1/9}{4} + \frac{1/9}{4}} = 0.964 \]

Thus, the \( t \) statistic is \( 3.492/0.964 = 3.622 \). We now compare this value with the \( t \) distribution values on 12 degrees of freedom (12 is equal to \( N - I \), the same number of degrees of freedom as \( s_p \)) in table D. We conclude that the p-value is between 0.001 and 0.0025. Thus, we reject the null hypothesis at the 5% level and conclude that there is some reduced growth due to the presence of nematodes.
(d) Define the contrast that compares the 0 treatment with the treatment with 10,000 nematodes. This contrast is a measure of the decrease in growth due to having a very large nematode infestation. Give a 95% confidence interval for this decrease in growth.

This contrast ignores $\mu_2$ and $\mu_3$ by giving each a zero coefficient. The contrast is

$$\Psi = \mu_1 - \mu_4$$

To give a confidence interval for $\Psi$, we use the corresponding sample statistic $c$ and its standard error:

$$c = \bar{x}_1 - \bar{x}_4 = 10.65 - 5.45 = 5.2$$

$$SE_c = s_p \sqrt{\frac{1}{4} + \frac{1}{4}} = 1.67 \sqrt{\frac{1}{2}} = 1.18$$

For a 95% confidence interval, the value of $t^*$ on 12 degrees of freedom is (from Table D) 2.179. Thus, the confidence interval is

$$5.20 \pm 2.179(1.18) = 5.20 \pm 2.571.$$ 

Page 493, exercise 6.64 Example 6.16 discusses a test about the mean contents of cola bottles. The hypotheses are

$$H_0 : \mu = 300$$

$$H_a : \mu < 300$$

The sample size is $n = 6$ and the population is assumed to have a normal distribution with $\sigma = 3$. A 5% significance test rejects $H_0$ if $z \leq -1.645$, where the test statistic is

$$z = \frac{\bar{x} - 300}{3/\sqrt{6}}.$$ 

Power calculations help us see how large a shortfall in the bottle contents the test can be expected to detect.

(a) Find the power of this test against the alternative $\mu = 299$.

As stated above, this test rejects $H_0$ whenever

$$\frac{\bar{x} - 300}{3/\sqrt{6}} \leq -1.645.$$ 

Solving for $\bar{x}$, we see that the test rejects $H_0$ whenever $\bar{x} \leq 297.99$. Now let’s assume that $\mu = 299$. The power may be defined as the probability that we will reject $H_0$ under this assumption, which is the same as the probability that $\bar{x}$ will be less than 297.99 under this assumption. Thus, we use Table A to compute

$$P(\bar{x} \leq 297.99) = P \left( \frac{\bar{x} - 299}{3/\sqrt{6}} \leq \frac{297.99 - 299}{3/\sqrt{6}} \right) = P(z \leq -0.82) = 0.2061.$$
Thus, the power is 0.2061.

(b) Find the power against the alternative $\mu = 295$.

The same reasoning as in part (a) shows that the power against $H_a : \mu = 295$ equals

$$P(\bar{x} \leq 297.99) = P \left( \frac{\bar{x} - 295}{3/\sqrt{6}} \leq \frac{297.99 - 295}{3/\sqrt{6}} \right) = P(z \leq 2.44) = 0.9927.$$  

Notice that we compute the same probability as in (a), namely $P(\bar{x} \leq 297.99)$, but we do it under a different assumption, namely that the true mean $\mu$ equals 295.

(c) Is the power against $\mu = 290$ higher or lower than the value you found in (b)? Explain why this result makes sense.

We may compute the power directly using the same logic as in (a) and (b) but this time under the assumption that $\mu = 290$ is the true mean:

$$P(\bar{x} \leq 297.99) = P \left( \frac{\bar{x} - 290}{3/\sqrt{6}} \leq \frac{297.99 - 290}{3/\sqrt{6}} \right) = P(z \leq 6.52) \approx 1.$$  

Notice that the probability is so far off the scale it’s essentially one. This means that we’re certain to detect a mean difference as large as 10 using a sample of size $n = 6$ (our power is nearly perfect). Clearly this implies that the power against $H_a : \mu = 290$ is higher than the power against $H_a : \mu = 295$. This makes intuitive sense because we would expect that the test should be more powerful against an alternative farther from the mean. In other words, if the truth were actually $\mu = 290$, we’d be more likely to detect a difference with our test than if the truth were $\mu = 295$. 