Solutions to Practice Problems 2

1. Suppose X has normal distribution with mean 0 and variance 1. Find $E(X^5)$, $P(X^3 > 0)$, and write down the density of $Y = 2X - 1$.

   Solution: As the density of X is symmetric around zero, $P(X^3 > 0) = P(X > 0) = \frac{1}{2}$. Since $x^5 e^{-\frac{1}{2}x^2}$ is an odd function of x, $E(X^5) = 0$. Y has normal distribution with mean $E(2X-1) = -1$ and variance $Var(2X-1) = 4$. So the density of Y is given by

   $$f_Y(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}(x+1)^2}.$$

2. If X is uniformly distributed on $(0,1)$, then find the distribution of $Y = -\log X$. Also find its hazard rate.

   Solution: $f_X(x) = 1$ if $0 < x < 1$ and 0 otherwise. Let $g(x) = -\log x$. $y = -\log x$ implies $x = e^{-y}$, so $g^{-1}(y) = e^{-y}$ for $y > 0$. Use the formula $f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$ if $y = g(x)$ for some x and 0 otherwise, to get $f_Y(y) = e^{-y}$ if $y > 0$ and zero otherwise. So Y is distributed as exponential with $\lambda = 1$. The hazard rate is $(f_Y(y)/P(Y > y)) = 1$.

3. Let X and Y be independent exponential random variables with the same mean. Find the density of X/Y.

   Solution: If mean of X is $1/\lambda$, $X\lambda$ and $Y\lambda$ both have exponential distribution with mean 1, and $X/Y = (X\lambda)/(Y\lambda)$, without loss of generality we may assume that $\lambda = 1$. For $a > 0$,

   $$F_{X/Y}(a) = P((X/Y) \le a) = \int_0^\infty \int_0^{ay} e^{-(x+y)}\, dx\, dy = \int_0^\infty (1 - e^{-ay})e^{-y}\, dy = a/(a+1).$$

   So $f_{X/Y}(a) = F'_{X/Y}(a) = (1+a)^{-2}$, $a > 0$.

4. Let X_1, X_2, X_3 are independent random variables uniformly distributed on $(-1,1)$. Find the distribution of $M = \max(X_1, X_2, X_3)$.

   Solution: $P(M \le a) = P(X_1 \le a)\, P(X_2 \le a)\, P(X_3 \le a) = (P(X_1 \le a))^3 = ((1+a)/2)^3$, if $-1 < a < 1$, $P(M \le a) = 0$ for $a \le -1$, and $P(M \le a) = 1$ for $a \ge 1$.

5. Two trucks break down at points randomly distributed on a road of length one mile. Assuming independence, find the expected value of the square of the distance between them.

   Solution: Suppose the one mile stretch starts at the point A. Let X and Y denote the distance from A to the points where the two trucks broke down. It is given that X and Y are independent and uniformly distributed on $(0,1)$. So

   $$E(X) = E(Y) = \frac{1}{2},\ E(X^2) = E(Y^2) = \int_0^1 t^2 dt = \frac{1}{3}.$$

$|X-Y|$ is the distance between the two trucks, $Cov(X,Y)=0$ and $E(X-Y)$. So

$$\begin{aligned} E(X-Y)^2 =& Var(X-Y) = Var(X)+Var(Y)-2Cov(X,Y) \\ =& Var(X)+Var(Y) = 2\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{6}. \end{aligned}$$

6. Suppose (X,Y) has bivariate normal distribution with correlation coefficient $\rho=-0.2378$. If X and Y have the same marginal distributions, find the correlation of $X+Y$ and $X-Y$.

   Solution: As $E(X-Y)=E(X)-E(Y)=0$ and $E(X^2)=E(Y^2)$, we have

   $$Cov(X+Y,X-Y) = E(X+Y)(X-Y) = E(X^2-Y^2) = 0.$$

   So $Corr(X+Y,X-Y)=0$.

7. The joint probability mass function of x and Y be given by $P_{X,Y}(x,y)=cxy$ if $x=1,2,4$; $y=1,3$ and 0 otherwise. Find c and $P(Y<X)$.

   Solution: $1=c(1+2+4)(1+3)=28c$. So $c=1/28$.
   $P(Y<X)=P_{X,Y}(2,1)+P_{X,Y}(4,1)+P_{X,Y}(4,3)=(2\times 1+4\times 1+4\times 3)/28=9/14$.

8. The joint probability density function of X and Y is given by

   $$f_{X,Y}(x,y)=\begin{cases} cxe^{-x^3} & \text{if} \quad -x<y<x, \quad x>0 \\ 0 & \text{otherwise,} \end{cases}$$

   for some $c>0$. Are X and Y independent? If they are not independent find the conditional density of Y given $X=2$. Justify your answer.

   Solution: X and Y are not independent as the range of Y depends on X. The marginal density f_X of X is given by

   $$f_X(x) = \int_{-x}^{x} cxe^{-x^3}\,dy = 2c\,x^2e^{-x^3},$$

   if $x>0$ and zero otherwise. So

   $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2x} \quad \text{if} \quad x>0, -x<y<x,$$

   and zero otherwise. Thus $f_{Y|X}(y|2)=\frac{1}{4}$ if $-2<y<2$ and zero otherwise. That is, given $X=2$, Y is uniformly distributed on $(-2,2)$.

9. Suppose the distribution of X is exponential. Find the correlation between $X+1$ and $-5X+2$.

   Solution:

   $$\begin{aligned} Cov(X+1,-5X+2) &= -5\times Var(X), \\ Var(-5X+2) &= 5^2\,Var(X) \\ Var(X+1) &= Var(X). \end{aligned}$$

So

$$Corr(X+1,-5X+2) = \frac{Cov(X+1,-5X+2)}{\sqrt{Var(X+1)\,Var(-5X+2)}} = -1.$$

10. Suppose X, Y are independent normal random variables with mean 0 and variance 1. Find the joint density of $(X-2Y,\ 2X+Y)$. Are $X-2Y$ and $2X+Y$ independent? Justify your answer.

Solution: Put $U = X-2Y$ and $V = 2X+Y$. So $X = \frac{1}{5}(U+2V)$ and $Y = \frac{1}{5}(V-2U)$. Jacobian is 5.

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi 5}\exp\left\{-\frac{1}{2}\left(\frac{u+2v}{5}\right)^2 - \frac{1}{2}\left(\frac{v-2u}{5}\right)^2\right\} \\ &= \frac{1}{\sqrt{2\pi 5}}\,e^{-\frac{1}{10}u^2}\,\frac{1}{\sqrt{2\pi 5}}\,e^{-\frac{1}{10}v^2}. \end{aligned}$$

So U and V are independent normal variables.

11. A fair coin is tossed 10 times, and suppose Y denotes the total number of times heads turned up. Let X denote the number of tails among the first 5 tosses. Find $Var(Y|X=0)$. (*Hint*: Check if X and $Y-5+X$ are independent).

Solution: As the number of head in the first 5 tosses $= 5-X$, it follows that $W = Y-(5-X)$ (= the number of heads in the last 5 tosses) is independent of X. Now W has binomial distribution with $n=5$ and $p=\frac{1}{2}$.

$$\begin{aligned} Var(Y|X=0) &= Var(Y-5+0|X=0) = Var(Y-5+X|X=0) \\ &= Var(W|X=0) = Var(W) = 5\times\frac{1}{2}\times\frac{1}{2} = 1.25. \end{aligned}$$

12. Suppose that the expected number of accidents per week at an industrial plant is 5. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 3. If the number of workers injured in each accident is independent of the number of accidents that occur, compute the expected number of workers injured in a week.

Solution: Let N = number of accidents in a week. $E(N) = 5$.

Let X_i = number of workers injured in i-th accident. $E(X_i) = 3$.

$S = \sum_{i\le N} X_i$ = number of workers injured in a week.

$$E(S|N=n) = E\left(\sum_{i=1}^{n} X_i|N=n\right) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = 3n.$$

$$E(S) = E(E(S|N)) = E(3N) = 3\times 5 = 15.$$