Practice problems

1. Rank the following in the order of most likely to least likely to occur. (You must compute the probabilities!)

   (a) A heart is drawn from an ordinary deck of cards.

   (b) A card is selected from an ordinary deck of cards and it is not a heart. In addition, a red ball is selected from an urn containing 3 red and 6 white balls. (Assume necessary independence).

   (c) On two independent tosses of a fair coin, tails are obtained on both tosses.

   Solution:

   (a) There are 13 hearts out of 52 cards. So the required probability is $\frac{1}{4}$.

   (b) Probability of not drawing a heart is $\frac{3}{4}$, and probability of drawing a red ball is $\frac{3}{9} = \frac{1}{3}$. By independence, the probability of selecting a red ball and not drawing a heart is $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$.

   (c) As all four outcomes $\{HH, HT, TH, TT\}$ are equally likely, the probability of two tails in two tosses of a fair coin is $\frac{1}{4}$.

   Thus all the three events are equally likely as the they all have the same probability $\frac{1}{4}$.

2. In how many ways can 4 novels, 5 mathematics books, and 1 chemistry book be arranged on a bookshelf if

   (a) the mathematics books must be together and the novels must be together;

   (b) the novels must be together but the other books can be arranged in any order?

   Solution:

   (a) The three subjects, math, novels and chemistry, can be arranged in 3! ways. Novels can be arranged among themselves in 4! ways and math books in 5! ways. So the total number of ways of arrangements is $3! \times 4! \times 5! \times 1! = 17,280$.

   (b) By treating the four novels as a single unit, the total of 7 units (one unit for novels, 5 math books, and one chemistry book) can be arranged among themselves in 7! ways. As the novels can be arranged among themselves in 4! ways, the total number of arrangements is $7! \times 4! = 120,960$.

3. Suppose $P(A) = 0.6, P(B) = 0.8, P(C) = 0.01$, $C \subset B$ and $P(A \cup B \cup C) = 0.82$. Then is it possible to conclude if A and B are independent. Justify your answer?

   Solution: $C \subset B$ implies $A \cup B \cup C = A \cup B$ and

   $$0.82 = P(A \cup B \cup C) = P(A \cup B) = P(A) + P(B) - P(AB) = 0.6 + 0.8 - P(AB)$$

   A and B are not independent, since $P(AB) = 0.58 \neq 0.6 \times 0.8 = P(A)P(B)$.

4. If E and F are independent then show that E and F^c are independent.

Solution: EF^c and EF are mutually exclusive and $E = EF^c \cup EF$. As E and F are independent, $P(EF) = P(E)P(F)$. So

$$P(EF^c) = P(E) - P(EF) = P(E) - P(E)P(F) = P(E)(1 - P(F)) = P(E)P(F^c)$$

5. At a certain school, 60% of the students wear neither a ring nor a necktie. 20% wear a ring and 30% wear a necktie. If one of the students is chosen randomly, what is the probability that this student is wearing a ring and a necktie?

Solution: Let R denote the event that the student wears a ring, and let N denote the event that the student wears a necktie. It is given that $P(R) = 0.2,\ P(N) = 0.3, P(R^c \cap N^c) = 0.6$. Since $P(R \cup N) = P(R) + P(N) - P(R \cap N)$, and $P(R \cup N) = 1 - P(R^c \cap N^c) = 0.4$, the required probability $P(R \cap N) = P(R) + P(N) - P(R \cup N) = 0.2 + 0.3 - 0.4 = 0.1$.

6. Of the registered voters, 35% are Democrats, 34% are Republican, and remaining are Independents. During the current health care reform debate, 70% of Democrats, 20% of the Republicans, and 50% of Independents prefer Public Option. Find the probability that a randomly chosen registered voter is a Democrat, given that the person prefers Public Option.

Solution: Let R = Republican, D = Democrat, I = Independent, and H = prefers Public Option. By the Bayes formula,

$$\begin{aligned} P(D \mid H) &= \frac{P(H \mid D)P(D)}{P(H \mid D)P(D) + P(H \mid R)P(R) + P(H \mid I)P(I)} \\ &= \frac{0.7 \times 0.35}{0.7 \times 0.35 + 0.2 \times 0.34 + 0.5 \times 0.31} = \frac{245}{468} = 0.52350427. \end{aligned}$$

7. Suppose that 5% of the men and 2% of the women working for a corporation make over \$200,000 a year. If 30% of the employees of the corporation are women, what percent of these who make over \$200,000 a year are women?

Solution: E = earn over \$200,000, M = men, W = women. $P(E|W) = 0.02, P(E|M) = 0.05, P(W) = 0.3$. The required percentage is $P(W|E) \times 100\%$, where

$$P(W|E) = \frac{P(E|W)P(W)}{P(E|W)P(W) + P(E|M)P(M)} = \frac{0.02 \times 0.3}{0.02 \times 0.3 + 0.05 \times 0.7} = \frac{6}{41}$$

8. Consider the function $f(x) = C(2x^2 - x^3)$ if $0 \le x \le 2$ and 0 otherwise. Could f be a probability density function? If so, determine C. Repeat if f were given by $f(x) = C(x-1)e^{x^2}$ for $0 \le x \le 2$ and 0 otherwise.

Solution: As $1 = \int_0^2 C(2x^2 - x^3)dx = C(\frac{16}{3} - 4) = \frac{4}{3}C$, it is a probability density function if $C = \frac{3}{4}$. As $f(0) = -C$ and $f(2) = Ce^4$, there is no value of C for which $f(x)$ is always non-negative. So f cannot be a probability density function.

9. A bus travels between the two cities A and B, which are 100 miles apart If the bus has breakdown, the distance from the breakdown to city A has a uniform distribution over $(0, 100)$. Three service stations are located at a distance of 25, 50, and 75 miles respectively, from A. Find the expected distance the bus needs to be towed.

   Solution: Let X denote distance from A to the breakdown site, and let L denote the distance from the breakdown site to the nearest service station. So,

$$L = \begin{cases} 25 - X, & \text{if } 0 < X < 25 \\ X - 25, & \text{if } \ 25 \leq X < 37.5 \\ |X - 50| & \text{if } 37.5 \leq X < 62.5 \\ 75 - X & \text{if } 62.5 \leq X < 75 \\ X - 75, & \text{if } \ 75 \leq X \leq 100. \end{cases}$$

$$E(L) = \frac{2}{100} \int_0^{25} x dx + \frac{4}{100} \int_0^{12.5} x dx = \frac{25}{4} + \frac{12.5}{4} = 9.375$$

10. The number of years a radio functions is exponentially distributed with parameter $\lambda = 0.5$. If Jones buys a used radio, what is the probability that it will be working after an additional four years?

    Solution: By the memoryless property of the exponential distribution, the required probability is $e^{-4\lambda} = e^{-2}$.

11. A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Suppose X denotes the number of defective items selected. Find $E(X)$. What is the name of the distribution of the random variable X?

    Solution: The distribution of X is hypergeometric.

$$\begin{aligned} E(X) =& 1 \times P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3) \\ =& \frac{\binom{4}{1}\binom{16}{2} + 2 \times \binom{4}{2}\binom{16}{1} + 3 \times \binom{4}{3}}{\binom{20}{3}} = \frac{57}{95} = \frac{3}{5} \end{aligned}$$

12. Let X be a geometric random variable with $p = 0.3$. Can you claim that

$$P(X > n + m | X > m) = P(X > n)$$

    for nonnegative integers m and n? If your answer is yes prove it, otherwise give your reasons.

    Solution: For any positive integer k, $P(X > k) = (1 - p)^k$, so

$$\begin{aligned} P(X > n + m | X > m) &= \frac{P((X > n + m) \cap (X > m))}{P(X > m)} = \frac{P(X > n + m)}{P(X > m)} \\ &= \frac{(1 - p)^{m+n}}{(1 - p)^m} = (1 - p)^n = P(X > n). \end{aligned}$$

13. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with mean $\mu = 71$ and variance $\sigma^2 = 6.25$. What percentage of 25-year-old men are over 6 feet tall? What percentage of men in the 6-footer club are over 6 feet 5 inches?

Solution: Let X denote the height, and Z denote a standard normal random variable.

$$P(X > 72) = P\left(\frac{X-\mu}{\sigma} > \frac{72-71}{2.5}\right) = P(Z > 0.4) = 1 - 0.6554 = .3446$$

$$P(X > 77|X > 72) = P\left(\frac{X-\mu}{\sigma} > \frac{77-71}{2.5}\right)/0.3446 = P(Z > 2.4)/0.3446$$
$$= (1 - 0.9918)/.3446 = .0082/.3446 = .0238$$

14. The cumulative distribution function of a random variable X is given by

$$F_X(a) = \begin{cases} 0, & \text{if } a < -2 \\ \frac{1}{2}, & \text{if } \ -2 \le a < 2 \\ \frac{2}{3}, & \text{if } \ 2 \le a < 3 \\ \frac{5}{6}, & \text{if } \ 3 \le a < 5 \\ 1, & \text{if } \ a \ge 5. \end{cases}$$

Determine the probability mass function of X. Compute $P(1 < X \le 4)$.

Solution: The probability mass function is given by

$$P(X=-2) = \frac{1}{2},\ P(X=2) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6},\ P(X=3) = \frac{5}{6} - \frac{2}{3} = \frac{1}{6},\ P(X=5) = 1 - \frac{5}{6} = \frac{1}{6}.$$

Note that

$$P(1 < X \le 4) = P(X=2) + P(X=3) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

15. The number of years a radio functions is exponentially distributed with parameter $\lambda = 0.25$. If Jones buys a used radio, what is the probability that it will be working after an additional four years?

Solution: By the memoryless property of the exponential distribution, the required probability is $e^{-4\lambda} = e^{-1}$.

16. Let the density of the random variable X be given by

$$f(a) = e^{-\pi(1-a)^2}, \quad -\infty < a < \infty.$$

(a) Identify the distribution of X. Write down the mean and variance of X.

(b) Find $P(X < 1)$.

Solution:

(a) Since

$$f(a) = \frac{1}{\sqrt{2\pi}\left(\sqrt{1/2\pi}\right)} e^{-\frac{1}{2(1/2\pi)}(a-1)^2}, \quad -\infty < a < \infty,$$

So the mean is 1, and variance is $\frac{1}{2\pi}$.

(b) Find $P(X < 1) = 0.5$.