Students will be able to:

- Describe Stochastic process;
- work with Poisson process.

Quiz 10.6 on page 368: $N(t)$ is a Poisson process of rate λ. $N'(t)$ counts only even-numbered arrivals of the process $N(t)$. Is $N'(t)$ a Poisson process? No.

$$P(N'(t)=k) = P(2k \le N(t) < 2k+2) = e^{-t\lambda}\frac{(t\lambda)^{2k}}{2k!}\left(1+\frac{t\lambda}{2k+1}\right)$$

Brownian Motion is also known as Wiener process (continuous-value continuous-time process).

- $W(0)=0$
- $W(t+s)-W(t)$ has normal distribution with mean zero and variance $s\alpha$
- $W(t+s)-W(t)$ is independent of $W(v)$ for all $v \le t$

BM has independent increments as in the case of Poisson process.

For $0 < t_1 < t_2 < \cdots < t_n$,

$$W(t_1), W(t_2)-W(t_1), \ldots, W(t_n)-W(t_{n-1})$$

are independent.
$W = (W(t_1), W(t_2), \ldots, W(t_n))$ has multivariate normal distribution.

Quiz 10.7 on page 369: $W(t)$ is a BM with variance $Var(W(t) = t\,\alpha$.
$X(t) = W(t)/\sqrt{\alpha}$ is a BM with $Var(X(t)) = t$

Nov. 28 10.7 Brownian Motion Process

Problem 10.7.1 on page 392: $X(0)$ is the opening price at the morning bell. t is hours. Exchange is open for 8 hours, and the standard deviation of the daily price change (the difference between the opening and closing bell prices) is 0.5 point. What is the value of the Brownian motion parameter α?

$$8\alpha = Var(X(8)) = 0.5^2, \quad \alpha = 1/32.$$

The function μ_X given by

$$\mu_X(t) = E[X(t)]$$

is called the *expected* value of the stochastic process $\{X(t)\}$

For Poisson Process N, the expected value $\mu_N(t) = t\lambda$
Similarly, for Brownian Motion X, $\mu_X(t) = 0$

Nov. 28 10.8 Expected value and correlation

Autocovariance function

$$C_X(t,s) = \mathrm{Cov}\,[X(t), X(t+s)], \quad C_X(m,k) = \mathrm{Cov}\,[X_m, X_{m+k}]$$

Autocorrelation function

$$R_X(t,s) = E[X(t)\,X(t+s)], \qquad R_X(m,k) = E[X_m\,X_{m+k}]$$

Clearly, $C_X(t,s) = R_X(t,s) - \mu_X(t)\mu_X(t+s)$

and, $C_X(m,k) = R_X(m,k) - \mu_X(m)\mu_X(m+k)$

For Poisson Process N, use the independent increments property to get
$C_N(t,s) = \mathrm{Cov}\left[N(t), N(t+s) - N(t) + N(t)\right] = \mathrm{Var}\left(N(t)\right) = \lambda\, t$
$R_N(t,s) = C_N(t,s) + \mu_N(t)\mu_N(t+s)) = \lambda\, t + \lambda^2 t(t+s)$, for $t, s \geq 0$

Similarly, for Brownian Motion X,
$C_X(t,s) = \mathrm{Cov}\left[X(t), X(t+s) - X(t) + X(t)\right] = \mathrm{Var}\left(X(t)\right) = \alpha\, t$
$R_X(t,s) = C_X(t,s) = \alpha\, t$ for $s > 0$. $\mathrm{Cov}\left[X(t), X(s)\right] = \min(t,s)\alpha$.

Students will be able to:

- Describe Stochastic process;
- work with Poisson process;
- recognize and apply Brownian Motion;
- derive expectation function, correlation and covariance functions.