Dec. 5 Desired outcomes from last class

Students will be able to:

- state Markov chain;
- work with n-step transition probabilities;
- use Chapman-Kolmogorov equations.

Dec. 5 12.2 Discrete-Time Markov Chain Dynamics

n-step transition probabilities $P_{ij}(n) = P[X_{n+m} = j | X_m = i]$.
Chapman-Kolmogorov equations.

$$P_{ij}(n+m) = \sum_k P_{ik}(n) P_{kj}(m), \quad \mathbf{P}(n+m) = \mathbf{P}(n)\mathbf{P}(m).$$

For a finite Markov chain, $\mathbf{P}(n) = \mathbf{P}^n$.

Dec. 5 12.2 Discrete-Time Markov Chain Dynamics

Find $\mathbf{P}(n)$ for

- $P_{01} = P_{10} = 1,\ P_{00} = P_{11} = 0.$
- $P_{01} = p,\ P_{10} = q,\ P_{00} = 1 - p,\ P_{11} = 1 - q.$
- $P_{01} = P_{10} = 0,\ P_{00} = P_{11} = 1.$

Dec. 5 12.3 Limiting state probabilities for a finite Markov chains

- For many Markov chains, $P_{ij}(n) \to \pi_j$ as $n \to \infty$.
  The limit does not depend on the initial state i.
- Such Markov chains are called ergodic Markov chains
- π_j are called stationary probabilities
- For a finite Markvo chain with state space {0, . . . , M}, a sufficient condition for ergodicity is that for some $r \geq 1$,

  $P_{ij}(r) > 0$ and for all $i = 0, \ldots, M$

Dec. 5 12.3 Limiting state probabilities for a finite Markov chains

For a Markov chain X_n, $P(X_n = j) = \sum_i P_{ij}(n)P(X_0 = i)$.
For ergodic Markov chains, the Chapman-Kolmogorov equations yield

$$\pi_j = \sum_k \pi_k P_{kj}. \qquad (*)$$

In fact π_j can be obtained by solving equation $(*)$ and $\sum_j \pi_j = 1$.
If all the rows of the transition probability matrix are identical, then $\mathbf{P}^n = \mathbf{P}$ for all $n \geq 1$. In this case π_j is the common value of P_{ij}.

Dec. 5 12.3 Limiting state probabilities for a finite Markov chains

Quiz 12.2 on page 451:

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0 & 0.6 & 0.4 \end{bmatrix}, \qquad \mathbf{P}^2 = \begin{bmatrix} 0.28 & 0.6 & 0.12 \\ 0.2 & 0.6 & 0.2 \\ 0.12 & 0.6 & 0.28 \end{bmatrix}.$$

$\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}, \quad \pi_1 = 0.2,\ \pi_2 = 0.6,\ \pi_3 = 0.2.$

Dec. 5 12.3 Limiting state probabilities for a finite Markov chains

Quiz 12.3 on page 454:
We can reach any state from any state with positive probability in four steps.

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{P}^2 = \begin{bmatrix} 0.81 & 0.09 & 0.1 \\ 1 & 0 & 0 \\ 0.9 & 0.1 & 0 \end{bmatrix}, \quad \mathbf{P}^4 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}.$$

$\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}, \quad \pi_1 = 5/6,\ \pi_2 = 1/12,\ \pi_3 = 1/12.$

Dec. 5 Desired outcomes

Students will be able to:

- use Chapman-Kolmogorov equations;
- understand ergodic Markov chains;
- find limiting state probabilities.