Exam 2 Solutions STAT/MATH 418 Fall 2011

1. Suppose X is uniformly distributed over $[-1,3]$ and $Y = X^2$. Find the cumulative distribution function (CDF) and the probability density function (PDF) of Y.

   Solution: For $y > 0$, the CDF is given by

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx.$$

   For $0 \le y \le 1$, $F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4}\,dx = \frac{1}{2}\sqrt{y}$. As $-1 \le X \le 3$, we have $F_Y(y) = \int_{-1}^{\sqrt{y}} \frac{1}{4}\,dx = \frac{1}{4}(1+\sqrt{y})$ for $1 \le y \le 9$. Further $F_Y(y) = 0$ for $y < 0$, and $F_Y(y) = 1$ for $y \ge 9$. The PDF of Y is obtained by differentiating F_Y,

$$F'_Y(y) = f_Y(y) = \begin{cases} 1/(4\sqrt{y}) & \text{if } 0 \le y \le 1 \\ 1/(8\sqrt{y}) & \text{if } 1 < y \le 9 \\ 0 & \text{otherwise.} \end{cases}$$

2. Find c, and $Cov(X,Y)$, where the joint probability density function of (X,Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} cx^2 & \text{if } |x| < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Are X and Y independent? Justify your answer.

   Solution: Note that $c\int_0^1 (\int_{-y}^{y} x^2 dx)dy = c\int_0^1 (2/3)y^3 dy = c\,(1/6)$. Hence $c = 6$. Random variables X and Y are not independent as $P(|X| \le Y) = 1$. For any odd integer k, $\int_{-y}^{y} x^k dx = 0$ for $0 < y < 1$. So $E(X) = E(XY) = 0$. Hence

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0.$$

3. The PDF of random variable and X and the conditional PDF of random variable Y given X are respectively,

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_{Y|X}(y|x) = \begin{cases} 2y/x^2 & \text{if } 0 \le y \le x,\ 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Find the joint PDF of (X,Y), and $E(X|Y = 1/2)$.

   Solution: The joint PDF is given by

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = \begin{cases} 6y & \text{if } 0 \le y \le x,\ 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   As $f_Y(1/2) = \int_{1/2}^{1} 6 \times \frac{1}{2}\,dx = \frac{3}{2}$,

$$f_{X|Y}(x|1/2) = \frac{f_{X,Y}(x,1/2)}{f_Y(1/2)} = \begin{cases} 6(1/2)/(3/2) = 2 & \text{if } \frac{1}{2} \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

   we have $E(X|Y = 1/2) = \int_0^1 x f_{X|Y}(x|1/2)dx = \int_{1/2}^{1} 2x dx = \frac{3}{4}$.

4. Let J and K be independent random variables with probability mass functions

$$P_J(j) = \begin{cases} 0.2 & \text{if } \ j = 1, 3, \\ 0.6 & \text{if } \ j = 2, \\ 0 & \text{otherwise,} \end{cases} \qquad P_K(k) = \begin{cases} 0.5 & \text{if } \ k = -1, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the moment generating function M_{J+K} of $J + K$. From this derive $E[(J + K)^3]$.

Solution: $M_J(t) = 0.2e^t + 0.6e^{2t} + 0.2e^{3t}$, and $M_K(t) = 0.5e^{-t} + 0.5e^t$. As J and K are independent,

$$M_{J+K}(t) = M_J(t)M_K(t) = 0.1 + 0.3e^t + 0.2e^{2t} + 0.3e^{3t} + 0.1e^{4t}.$$

$M'''_{J+K}(t) = 0.3e^t + 0.2(2^3)e^{2t} + 0.3(3^3)e^{3t} + 0.1(4^3)e^{4t}$, So $E[(J + K)^3] = M'''_{J+K}(0) = 16.4$.

5. The number of people who enter an elevator on the ground floor, to go up in Thomas building, is Poisson random variable with mean 4. Thomas building has 5 floors (ignore the basement). Assume that no one enters at any of the other floors. If each person is equally likely to get off at any one of the floors $2, 3, 4, 5$, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passengers.

   Hint: Define $I_i = 1$ or 0 according as the elevator stops at floor i or not; and N the number of people entering the elevator on the ground floor. You may use $E(X) = E[E(X|N)]$.

   Solution: Let $I_i = 1$ or 0 according as the elevator stops at floor i or not. Let N denote the number of people entering the elevator on the ground floor. The distribution of N is given to be Poisson with mean 4. For $k \geq 1$, we have $E\left(\sum_{i=2}^{5} I_i | N = k\right) = \sum_{i=2}^{5} E(I_i | N = k) = 4(1-(3/4)^k)$, and $E\left(\sum_{i=2}^{5} I_i | N = 0\right) = 0$. So $E\left(\sum_{i=2}^{5} I_i\right) = \sum_{k=0}^{\infty} 4(1-(3/4)^k)e^{-4}4^k(1/k!) = 4(1 - e^{-1})$.