Practice problems for the Stat 418 final exam – Fall 2011

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1. What is the probability that at least one of a pair of fair dice lands on 6, given that the sum of the dice is 8?

2. A fair dice is rolled 100 times, and suppose Y denotes the total number of times five turned up. Find the moment generating function of Y.

3. Let X_1, X_2, X_3 be independent normal variables with means $0, 1, 2$ and variances $1, 4, 8$ respectively. Find the moment generating function of $Y = X_1 - 2X_2 + X_3$.

4. Suppose that X, Y, H are independent random variables, each being uniformly distributed over $(0, 1)$. Find $P(X \leq H, \text{ and } Y \leq H)$

5. Let X, Y be independent exponential random variables with mean 1. Find the joint density of $(X + Y, X - Y)$. Write down clearly the ranges for the variables.

6. Let the joint probability density of X, Y is given by

$$f_{X,Y}(x, y) = \begin{cases} Cxy & \text{if} \quad 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find C and the marginal density of X. Also find $P(2X \leq Y)$.

7. Suppose that the expected number of accidents per week at an industrial plant is 5. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 3. If the number of workers injured in each accident is independent of the number of accidents that occur, compute the expected number of workers injured in a week.

8. A fair coin is tossed 100 times, and suppose Y denotes the total number of times heads turned up. Let X denote the number of tails among the first 50 tosses. Find $Var(Y|X = 0)$. (*Hint*: Check if X and $Y - 50 + X$ are independent).

9. The joint probability density function of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} c\,x\,e^{-x^3} & \text{if} \quad -x < y < x, \quad x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

   for some $c > 0$. Find the conditional density of $Y|X = 1$. Compute $Var(Y|X = 1)$.

10. Suppose X is a continuous random variable with mean and variance both equal to 10.335. Find $Corr(10.335X + 17, -6X + 10.335)$.

11. At a car garage, there is always a backlog of car waiting to be serviced. The service times of cars are i.i.d. exponential random variables with a mean service time of 30 minutes. Find the probability mass function of $N(t)$, the number of cars serviced in the first t hours of the day.

12. A certain town's weather is classified each day as being rainy, sunny, or overcast, but dry. If it is rainy one day, then it is equally likely to be either sunny or overcast the following day. If it is not rainy, then there is one chance in three that the weather will persist in whatever state it is in for another day, and if it does change, then it is equally likely to become either of the other two states. In the long run what proportion of days are sunny? What proportion are rainy?

Solutions for the practice problems for the Stat 418 final exam

1. What is the probability that at least one of a pair of fair dice lands on 6, given that the sum of the dice is 8?

   Solution: Let A denote the event that at least one of the dice lands on 6, and B denote the event that the sum is 8. As $B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$, and $A \cap B = \{(2,6),(6,2)\}$, it follows that $P(A|B) = P(A \cap B)/P(B) = 2/5 = 0.4$.

2. A fair dice is rolled 100 times, and suppose Y denotes the total number of times five turned up. Find the moment generating function of Y.

   Solution: Y has binomial distribution with $n = 100$, and $p = 1/6$. As Y can be written as sum of 100 independent Bernoulli random variables with $p = 1/6$, the moment generating function M of Y is given by $M(t) = E(e^{tY}) = ((1/6)e^t + (5/6))^{100}$.

3. Let X_1, X_2, X_3 be independent normal variables with means $0, 1, 2$ and variances $1, 4, 8$ respectively. Find the moment generating function of $Y = X_1 - 2X_2 + X_3$.

   Solution: $E(Y) = 0$ and $Var(Y) = 1 + 4 \times 4 + 8 = 25$. Note that Y has normal distribution with mean zero and variance 25. Thus its moment generating function is given by $M(t) = e^{\frac{1}{2}25t^2} = e^{12.5t^2}$.

4. Suppose that X, Y, H are independent random variables, each being uniformly distributed over $(0,1)$. Find $P(X \le H, \text{ and } Y \le H)$

   Solution: As X and Y are independent, $P(X \le h, Y \le h) = P(X \le h)P(Y \le h)$ for all h. As H is independent of X, Y,

$$\begin{aligned} P(X \le H, \text{ and } Y \le H) &= \int_0^1 P(X \le H, Y \le H | H = h)\, dh \\ &= \int_0^1 P(X \le h, Y \le h | H = h)\, dh \\ &= \int_0^1 P(X \le h, Y \le h)\, dh \\ &= \int_0^1 P(X \le h)P(Y \le h)\, dh = \int_0^1 h^2\, dh = \frac{1}{3}. \end{aligned}$$

5. Let X, Y be independent exponential random variables with mean 1. Find the joint density of $(X+Y, X-Y)$. Write down clearly the ranges for the variables.

   Solution: Put $U = X + Y$, $V = X - Y$. So $2X = U + V$ and $2Y = U - V$. As X, Y are nonnegative valued random variables, $U > 0$ and $|V| \le U$ with probability 1. The joint density $f_{X,Y}$ of (X, Y) is given by $f_{X,Y}(x, y) = e^{-x-y}$ for $x, y > 0$ and 0 otherwise.

$$|det(J(x,y)| = \left| det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = 2.$$

So the joint density f of (U, V), is given by

$$f(u,v) = \begin{cases} \frac{1}{2}\, e^{-x-y} = \frac{1}{2}\, e^{-u} & \text{if } u > 0,\ |v| \leq u \\ 0 & \text{otherwise.} \end{cases}$$

6. Let the joint probability density of X, Y is given by

$$f_{X,Y}(x,y) = \begin{cases} Cxy & \text{if } \quad 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find C and the marginal density of X. Also find $P(2X \leq Y)$.

Solution: $\int_0^1 \int_0^y x\, y dx\, dy = \int_0^1 \frac{1}{2}\, y^3 dy = \frac{1}{8}$. So $C = 8$.

$P(2X \leq Y) = \int_0^1 \int_{2x \leq y} 8x\, y\, dx\, dy = \int_0^1 \frac{1}{2}(y/2)^2\, 8\, y\, dy = \int_0^1 y^3\, dy = \frac{1}{4}$.

7. Suppose that the expected number of accidents per week at an industrial plant is 5. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 3. If the number of workers injured in each accident is independent of the number of accidents that occur, compute the expected number of workers injured in a week.

Solution: Let N = number of accidents in a week. $E(N) = 5$.

Let X_i = number of workers injured in i-th accident. $E(X_i) = 3$.

$S = \sum_{i \leq N} X_i$ = number of workers injured in a week.

$$E\left(S|N = n\right) = E\left(\sum_{i=1}^{n} X_i | N = n\right) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E\left(X_i\right) = 3n.$$

$$E(S) = E(E(S|N)) = E(3N) = 3 \times 5 = 15.$$

8. A fair coin is tossed 100 times, and suppose Y denotes the total number of times heads turned up. Let X denote the number of tails among the first 50 tosses. Find $Var(Y|X = 0)$. (*Hint*: Check if X and $Y - 50 + X$ are independent).

Solution: As the number of heads in the first 50 tosses $= 50 - X$, $W = Y - (50 - X) =$ the number of heads in the last 50 tosses is independent of X. Now W has binomial distribution with $n = 50$ and $p = \frac{1}{2}$.

$$\begin{aligned} Var(Y|X = 0) &= Var(Y - 50 + 0|X = 0) = Var(Y - 50 + X|X = 0) \\ &= Var(W|X = 0) = Var(W) = 50 \times \frac{1}{2} \times \frac{1}{2} = 12.5. \end{aligned}$$

9. The joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c\,x\,e^{-x^3} & \text{if} \quad -x < y < x, \quad x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

for some $c > 0$. Find the conditional density of $Y|X = 1$. Compute $Var(Y|X = 1)$.

Solution: $f_X(x) = 2c\,x^2 e^{-x^3}$ if $x > 0$ and zero otherwise. So $f_{Y|X}(y|x) = \frac{1}{2x}$ if $x > 0, -x < y < x$ and zero otherwise. Thus $f_{Y|X}(y|1) = \frac{1}{2}$ if $-1 < y < 1$ and zero otherwise. $E(Y|X = 1) = 0$ and

$$E(Y^2|X = 1) = \int_{-1}^{1} \frac{1}{2} y^2\, dy = \frac{1}{3}.$$

So $Var(Y|X = 1) = \frac{1}{3}$.

10. Suppose X is a continuous random variable with mean and variance both equal to 10.335. Find $Corr(10.335X + 17, -6X + 10.335)$.

Solution:

$$\begin{aligned} Cov(10.335X + 17, -6X + 10.335) &= -6 \times 10.335\; Var(X), \\ Var(10.335X + 17) &= (10.335)^2\; Var(X) \\ Var(-6X + 10.335) &= 6^2\; Var(X). \end{aligned}$$

So

$$Corr(10.335X+17, -6X+10.335) = \frac{Cov(10.335X + 17, -6X + 10.335)}{\sqrt{Var(10.335X + 17)\; Var(-6X + 10.335)}} = -1.$$

11. At a car garage, there is always a backlog of car waiting to be serviced. The service times of cars are i.i.d. exponential random variables with a mean service time of 30 minutes. Find the probability mass function of $N(t)$, the number of cars serviced in the first t hours of the day.

Solution: Since there is always a backlog and the service times are i.i.d. exponential random variables, the time between service completions are a sequence of iid exponential random variables. that is, $N(t)$ is a Poisson process. Since the expected service time is 30 minutes, the rate of the Poisson process is $\lambda = 1/30$ per minute. Since t hours equals $60t$ minutes, the expected number serviced is $\lambda(60t)$ or $2t$. Thus the probability mass function of $N(t)$ is given by $P(N(t) = k) = (2t)^k e^{-2t}(1/k!)$, for $k = 0, 1, \ldots$

12. A certain town's weather is classified each day as being rainy, sunny, or overcast, but dry. If it is rainy one day, then it is equally likely to be either sunny or overcast the following day. If it is not rainy, then there is one chance in three that the weather will

persist in whatever state it is in for another day, and if it does change, then it is equally likely to become either of the other two states. In the long run what proportion of days are sunny? What proportion are rainy?

Solution: The successive weather classifications constitute a Markov chain. If the states are 0 for rainy, 1 for sunny, and 2 for overcast, then the transition probability matrix is given by:

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

The long run proportions satisfy

$$\begin{aligned} \pi_0 &= \pi_1(1/3) + \pi_2(1/3) \\ \pi_1 &= \pi_0(1/2) + \pi_1(1/3) + \pi_2(1/3) \\ \pi_2 &= \pi_0(1/2) + \pi_1(1/3) + \pi_2(1/3) \\ 1 &= \pi_o + \pi_1 + \pi_2 \end{aligned}$$

The solution of the system of equations is $\pi_0 = 1/4, \pi_1 = \pi_2 = 3/8$. Hence, three-eighths of the days are sunny and one-fourth are rainy.