Random Fractals with Stationary Scale Coefficients

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1 Introduction

Dimension-like characteristics of fractals obtained by random Cantor type recursive constructions have been studied in a number of mathematical and physical papers (see, e.g., [7],[16],[12],[15],[18],[21]). These constructions are specified by some qualitative geometric requirements as well as by random "scale coefficients" that, roughly speaking, characterize the degree of contraction of sets when going from one level to the next one. Among many dimensional characteristics of fractals, the Hausdorff dimension is the most important and widely used. In the above cited papers the Hausdorff dimension of the generated fractal is found as the unique solution of the respective "Moran equation".

Most attention has been devoted to the "memoryless" case when the scale coefficients do not depend on the "prehistory". A comprehensive study of the deterministic constructions with "memory" has been done by Barreira in [1].

Constructions with memory in a random context have been introduced and studied in [6], where it has been assumed that the scale coefficients at different levels are stochastically independent but not necessarily identically distributed.

In the present paper we study quite distinct random model with non-zero memory in which the scale coefficients are stationary connected. We calculate the Hausdorff dimension of the generated random fractals.

We consider random fractals in general complete random metric spaces. By the standard encoding procedure our study is reduced to the "leading special case": the space $X^+$ of finitely-valued one-sided sequences, endowed with a random "scale" metric compatible with the construction parameters. The coding maps the fractal onto $X^+$ and the sets of the construction level $n$ onto cylinders with base $\{1, \ldots, n\}$. We compute the "global" Hausdorff dimension by calculating the "local" cylinder-wise dimension of an appropriate random probability measure $\mu_\omega$ and employing [21, Theorem 2.1] that claims that these dimensions coincide. (Let us remind that the notion of cylinder-wise dimension was introduced in [2, 3, 4], [9] and then widely used by other authors: see, e.g., the citations above). Although this general scheme is the same as in [6], the results as well as the tools we use here (ergodic theorems, notions of statistical mechanics) differ significantly.

Let $(\Omega, \mathcal{F}, P)$ denote the probability space. The random measure $\mu_\omega$ on the space $X$ of finitely-valued two-sided sequences, for which we calculate the cylinder-wise "local" dimension, is a random Gibbs measure; its potential is specified by the random scale coefficients and by a non-random factor ("the inverse temperature") $\beta > 0$. For each $\beta$ we compute the pressure $P(\beta)$ which happens to be non-random. Let $\alpha$ be the (unique) solution of the Bowen equation $P(\beta) = 0$ (this equation can be also considered as a stochastic version of the Moran equation). The Hausdorff dimension of the set $X^+$ with respect to the scale metric $\rho_\omega$ almost surely equals $\alpha$. Up to Section 5 we study the special case when both the random metric space and the fractal set coincide with $(X^+, \rho_\omega)$ (the subsets of level $n$ in the recursive construction are cylinders with base $\{1, \ldots, n\}$). The general case of an arbitrary complete metric space is studied in Section 6.
In Section 2 we consider random analogs of the classical notions of the Statistical Mechanics: the random potential and the corresponding random Gibbs specification. Some auxiliary problems such as the existence and uniqueness of the specified Gibbs measure \( \mu_\omega \) are studied. As one of the tools we consider the skewed product of the measures \( \mu_\omega \) and \( P \) in the product space \( X \times \Omega \). Note that the same product of the measures has been used, for example, in [16].

In Section 3 we prove several limit theorems that are used in the sequel. It is shown that the random scale metric \( \rho_\omega \) is "averageable" in the sense of [21]. It is also proved that the pressure \( P(\beta) \) exists for every \( \beta > 0 \) and is non-random.

In Section 4 we study some properties of \( P(\beta) \). It is proved, in particular, that the number \( \alpha \) defined above exists. We also interpret \( P(\beta) \) as the exponential characteristic (the logarithm of the spectral radius) of some limit random matrix.

In Section 5 we prove that for almost all \( \omega \) the cylinder-wise dimension of the measure \( \mu_\omega \) and, therefore, the Hausdorff dimension of the set \( X \) with respect to the random scale metric \( \rho_\omega \), is equal to \( \alpha \), where \( \alpha \) satisfies \( P(\alpha) = 0 \).

In Section 6 we generalize the previous results to random fractals with "finite memory" in arbitrary complete random metric spaces \( (M, \rho_\omega) \). The Hausdorff dimension of such a fractal \( F(\omega) \) is found by establishing an isometry (coding) between the fractal \( F(\omega) \) (with the induced metric) and the space of one-sided sequences \( (X^+, \rho_\omega) \).

In general, solving the Bowen equation \( P(\beta) = 0 \) presents significant computational difficulties. In the final Section 7 we give several examples, involving periodical dynamical systems, where the pressure \( P(\beta) \) can be found explicitly.

\section{Random Scale Coefficients and Associated Random Gibbs Measures}

First we introduce some notations. Let \( K \) be the set of all finite subsets of \( \mathbb{Z} \). We put \( X := S^\mathbb{Z} \). Let \( V \subset \mathbb{Z} \) and \( x \in X \). By \( x_V \) we shall denote \( (x_i : i \in V) \), the restriction of \( x \) to the set \( V \), and by \( X_V \) the set of all such restrictions. In \( X \) we consider cylinders:

\[ C_V(x) := \{ y \in X : y_V = x_V \}, \quad x \in X, \ V \in K. \]

We denote by \( \mathcal{A} \) and \( T \) the \( \sigma \)-algebra and topology in \( X \) generated by the cylinders. Let \( \tau \) be the left-ward shift in \( X \), that is \( (\tau x)_i = x_{i+1} \) for any \( i \in \mathbb{Z} \) and \( x \in X \). Along with the space \( X \) we consider a probability space \( (\Omega, \mathcal{G}, P) \) with an ergodic endomorphism \( \sigma \).

Let \( m \in \mathbb{N} \). We shall consider real valued functions

\[ L(x_1, \ldots, x_m; \omega), \quad x_1, \ldots, x_m \in S, \ \omega \in \Omega. \]

We assume that \( L(x_1, \ldots, x_m; \omega) \) is a measurable function of \( \omega \) and

\[ 0 < L(x_1, \ldots, x_m; \omega) < 1, \quad \text{for all } x_1, \ldots, x_m \text{ and } \omega. \]
In Section 6 we study random fractals in general metric spaces obtained by iterated constructions. $L(x_1, \ldots, x_m; \omega)$ will play role of the scale coefficients of such constructions.

In case of need we shall impose the following conditions on the coefficients. Namely, the condition

$$E \log L(x_1, \ldots, x_m; \omega) > -\infty, \quad x_1, \ldots, x_m \in X,$$

and also the stronger condition

$$\text{ess inf}_{\omega \in \Omega} L(x_1, \ldots, x_m; \omega) > 0, \quad x_1, \ldots, x_m \in X. \quad (2.2)$$

The following notions of Statistical Mechanics will be useful in the sequel. For each $\omega \in \Omega$ we define the random potential $U$ as follows

$$U(\{i - m + 1, \ldots, i\}, x; \omega) := -\log L(x_{i-m+1}, \ldots, x_i; \sigma^i\omega),$$

and $U(V, x; \omega) = 0$, if $V \in K$ is not a set of $m$ contiguous elements of $\mathbb{Z}$.

We remind here the definition of the norm of the potential.

$$\|U(\cdot, \cdot, \omega)\| := \sup_{t \in \mathbb{Z}} \sum_{t \in A \in K} \max_x |U(A, x; \omega)|. \quad (2.3)$$

Note that if condition (2.2) is fulfilled then $\|U(\cdot, \cdot, \omega)\| < \infty$ for every $\omega \in \Omega$.

It is easy to see that this random potential $U$ is invariant in the following sense. For every $V \in K$, every $x \in X$, and every $\omega \in \Omega$

$$U(V, x; \omega) = U(V - 1, \tau x; \sigma\omega). \quad (2.4)$$

For $V \in K$ we denote $\partial V = \{i \in \mathbb{Z} : 0 < \min_{k \in V} |i - k| < m\}$, the "boundary" of $V$. In particular, when $V = \{k, \ldots, n\}$ we have

$$\partial V = \{k - m + 1, \ldots, k - 1, n + 1, \ldots, n + m - 1\}.$$

Let $V = \{k, \ldots, n\}$. For every $\omega \in \Omega$ and $x \in X$ we define the energy

$$E(x_V|x_{\partial V}; \omega) := \sum_{i=k}^{n+m-1} U(\{i - m + 1, \ldots, i\}, x; \omega)$$

$$= -\log \prod_{i=k}^{n+m-1} L(x_{i-m+1}, \ldots, x_i; \sigma^i\omega). \quad (2.5)$$

Let

$$a_V(x; \omega) := e^{-E(x_V|x_{\partial V}; \omega)} = \prod_{i=k}^{n+m-1} L(x_{i-m+1}, \ldots, x_i; \sigma^i\omega),$$

$$Z_V^{(\beta)}(x; \omega) := \sum_{x_V \in X_V} e^{-\beta E(x_V|x_{\partial V}; \omega)} = \sum_{x_V \in X_V} \prod_{i=k}^{n+m-1} L^{\beta}(x_{i-m+1}, \ldots, x_i; \sigma^i\omega).$$
The following equation defines the Gibbs specification for each \( \omega \in \Omega \) and \( \beta > 0 \):

\[
\gamma_{\omega}(x_{V} | x_{\partial V}) = \frac{(a_{V}(x; \omega))^{\beta}}{Z_{V}^{(\beta)}(x; \omega)}.
\] (2.6)

Next we establish the existence, uniqueness, and study some important properties of the random probability measure specified by \( \gamma_{\omega} \).

**Theorem 2.1** Assume that condition (2.1) is fulfilled. Then for almost all \( \omega \in \Omega \) there is a unique probability measure \( \mu_{\omega} \) on \( \mathcal{A} \) specified by \( \gamma_{\omega} \).

**Proof.** First, since for every \( \omega \) the specification \( \gamma_{\omega} \) is consistent, such probability measure on \( \mathcal{A} \) exists for every \( \omega \) (see, e.g., [19]).

By Rost’s theorem (see, e.g., [10, Theorem 8.39]), for the uniqueness it is enough to show that for almost all \( \omega \) for every \( V \in K \) and every \( x, y \in X \)

\[
|E(x_{V} | x_{\partial V}; \omega) - E(x_{V} | y_{\partial V}; \omega)| \leq C(\omega) < \infty,
\] (2.7)

where \( C(\omega) \) is independent of \( x, y \) and \( V \).

Now, from (2.1) it follows immediately that

\[
E \log \frac{L(y_{1}, \ldots, y_{m}; \omega)}{L(x_{1}, \ldots, x_{m}; \omega)} \leq
\]

\[
- E \log L(y_{1}, \ldots, y_{m}; \omega) - E \log L(x_{1}, \ldots, x_{m}; \omega) < \infty
\]

for any \( x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{m} \in S \). Assume that \( V = \{k, \ldots, n\} \) and \( x_{V} = y_{V} \). Then we have

\[
|E(x_{V} | x_{\partial V}; \omega) - E(x_{V} | y_{\partial V}; \omega)| \leq
\]

\[
\sum_{k \leq i \leq k+m-2, \atop n+1 \leq i \leq n+m-1} \left| \log \frac{L(y_{i} - m + 1, \ldots, y_{i}; \sigma^{i} \omega)}{L(x_{i} - m + 1, \ldots, x_{i}; \sigma^{i} \omega)} \right|
\] (2.9)

Thus (2.8) and (2.9) imply that

\[
E \sup_{V} \max_{x, y} |E(x_{V} | x_{\partial V}; \omega) - E(x_{V} | y_{\partial V}; \omega)| \leq
\]

\[
2(m - 1) E \max_{x, y} \left| \log \frac{L(y_{1}, \ldots, y_{m}; \omega)}{L(x_{1}, \ldots, x_{m}; \omega)} \right| < \infty.
\]

Therefore, for \( P \)-almost every \( \omega \) the inequality (2.7) holds with

\[
C(\omega) = \sup_{V \in K} \max_{x, y \in X} |E(x_{V} | x_{\partial V}; \omega) - E(x_{V} | y_{\partial V}; \omega)|.
\]

\[\square\]

**Lemma 2.1** For \( P \)-almost every \( \omega \in \Omega \) and each \( A \in \mathcal{A} \)

\[
\mu_{\omega}(\tau^{-1} A) = \mu_{\sigma_{\omega}}(A).
\] (2.10)
Proof. Fix $\omega \in \Omega$. By $\tilde{\mu}_\omega$ we denote the measure such that $\tilde{\mu}_\omega(A) = \mu_\omega(\tau^{-1}A)$ for every $A \in \mathcal{A}$. And by $\tilde{\gamma}_\omega$ denote the specification such that 
\[
\tilde{\gamma}_\omega(xV | x\partial V) = \gamma_\omega((\tau^{-1}x)Vg | (\tau^{-1}x)\partial Vg).
\]
It is easy to verify that the measure $\tilde{\mu}_\omega$ is specified by $\tilde{\gamma}_\omega$. On the other hand, (2.4) implies that $\tilde{\gamma}_\omega = \gamma_{\sigma \omega}$. Since, by Theorem 2.1 the probability measure $\mu_{\sigma \omega}$ specified by $\gamma_{\sigma \omega}$ is unique we find that for almost every $\omega$ the measure $\tilde{\mu}_\omega$ coincides with $\mu_{\sigma \omega}$. This finishes the proof. 

$\square$

Lemma 2.2 For every $A \in \mathcal{A}$ the function 
\[
\omega \mapsto \mu_\omega(A)
\]
is $\mathbf{P}$-measurable. \hspace{2cm} (2.11)

Proof. Fix $\omega \in \Omega$. We shall remind that for each $\omega$ a specified measure $\mu_\omega$ can be found as following [cf.]. Let $V_k$, $k = 1, 2, \ldots$ be an increasing sequence of subsets of $T$ such that $V_k \uparrow T$ as $k \to \infty$, and let $y \in X$. For every $k$ we define a measure $\mu^k_\omega$ by 
\[
\mu^k_\omega(C_{V_k}(x)) = \gamma_\omega(x_{V_k} | y_{V_k}).
\]
Then any limit point of $\{\mu^k_\omega\}_{k=1}^\infty$ is in $G(\gamma_\omega)$. If a specified probability measure $\mu_\omega$ is unique then $\{\mu^k_\omega\}_{k=1}^\infty$ converges to $\mu_\omega$ weakly. Therefore for any $W \in K$ and $x \in X$ we have 
\[
\mu_\omega(C_W(x)) = \lim_{k \to \infty} \mu^k_\omega(C_W(x)).
\]
Choose $k$ such that $W \subset V_k$. Then 
\[
\mu^k_\omega(C_W(x)) = \sum_{x_{V_k \setminus W} \in S_k \setminus W} \mu^k_\omega(C_{V_k}(x)) = \sum_{x_{V_k \setminus W} \in S_k \setminus W} \gamma_\omega(x_{V_k} | y_{V_k}).
\]
This implies that (2.11) holds when $A$ is a cylinder.

Now, let 
\[
\mathcal{B} = \{A \in \mathcal{A} : \omega \mapsto \mu_\omega(A) \text{ is measurable}\}.
\]
Claim: $\mathcal{B}$ is a monotone class. Indeed, let $A_1 \subset A_2 \subset \ldots$, where each $A_k \in \mathcal{B}$. And let $A = \cup_{k=1}^\infty A_k$. Then for each $\omega$ such that $\mu_\omega$ is defined 
\[
\mu_\omega(A) = \lim \mu_\omega(A_k).
\]
Hence $A \in \mathcal{B}$.

Similarly, if $A_1 \supset A_2 \supset \ldots$, where each $A_k \in \mathcal{B}$, then $\cap_{k=1}^\infty A_k \in \mathcal{B}$. Note that $\mathcal{B}$ contains the algebra of cylinders. Therefore, (see, e.g., [11]) the family $\mathcal{B}$ must contain the $\sigma$-algebra $\mathcal{A}$, which means $\mathcal{B} = \mathcal{A}$.

If $C \in \mathcal{A} \otimes G$ then for each $\omega \in \Omega$ we consider the section 
\[
C_\omega = \{x \in X : (x, \omega) \in C\}.
\]
It is known that $C_\omega \in \mathcal{A}$ for every $\omega \in \Omega$ (see, e.g., [11]).
Lemma 2.3 For every $C \in \mathcal{A} \otimes \mathcal{G}$ the function
\[ \omega \mapsto \mu_\omega(C_\omega) \] is $\mathbf{P}$-measurable. \hfill (2.12)

Proof. We shall use the same argument as in the proof of Lemma 2.2. Let
\[ C = \{ C \in \mathcal{A} \otimes \mathcal{G} : \omega \mapsto \mu_\omega(C_\omega) \text{is measurable} \} \]

By Lemma 2.2 the family $C$ contains the sets $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{G}$. We claim $C$ is a monotone class. Let $C_1 \subset C_2 \subset \ldots$, where each $C_k \in C$. And let $C = \cup_{k=1}^{\infty} C_k$. Then it is clear that for every $\omega$
\[ C_\omega \subset C_2 \omega \subset \ldots \text{ and } C_\omega = \cup_{k=1}^{\infty} C_k_\omega. \]

Since $\omega \mapsto \mu_\omega(C^k_\omega)$ is $\mathbf{P}$-measurable for every $k$, $\omega \mapsto \mu_\omega(C_\omega)$ is $\mathbf{P}$-measurable too. Similarly it can be shown that $C$ is closed under countable intersections of decreasing sequences of sets. Now, because the sets $A \times B$, $A \in \mathcal{A}, B \in \mathcal{G}$ generate $\mathcal{A} \otimes \mathcal{G}$, we conclude that $C = \mathcal{A} \otimes \mathcal{G}$ (see, e.g., [11]). □

Let $C \in \mathcal{A} \otimes \mathcal{G}$. We define the measure $Q$ in $\mathcal{A} \otimes \mathcal{G}$ by
\[ Q(C) := \int_\Omega \mu_\omega(C_\omega) \mathbf{P}(d\omega). \hfill (2.13) \]

The following statement is obvious.

Lemma 2.4 $Q(C) = 1$ if and only if $\mu_\omega(C_\omega) = 1$ for $\mathbf{P}$-almost every $\omega$.

In the product space $X \times \Omega$ we define shift $\tau \otimes \sigma$ by $(\tau \otimes \sigma)(x, \omega) := (\tau x, \sigma \omega)$, where $x \in X$ and $\omega \in \Omega$.

Lemma 2.5 Measure $Q$ is $(\tau \otimes \sigma)$-invariant.

Proof. Lemma 2.1 and [5, Lemma 2]. □

3 Limit Theorems

Let $V = \{1, \ldots, n\}$, where $n \geq 1$. We shall make frequent use of the following notations $C_n(x) := C_V(x)$, $a_n(x; \omega) := a_V(x; \omega)$, and $Z^{(j)}_{n}(x; \omega) := Z^{(j)}_{V}(x; \omega)$. If $n \geq m$ we also put
\[ \tilde{a}_n(x; \omega) := \prod_{i=m}^{n} L(x_{i-m+1}, \ldots, x_i; \sigma^i \omega). \]

Theorem 3.1 The following limit
\[ \lim_{n \to \infty} \frac{1}{n} \log a_n(x, \omega) \]
exists for $Q$-almost all $(x, \omega)$ and is $(\tau \otimes \sigma)$-invariant.
Proof. Let $x \in X$ and $\omega \in \Omega$. We have

$$\frac{1}{n} \log a_n(x, \omega) = \frac{1}{n} \sum_{i=1}^{n+m-1} \log L(x_{i-m+1}, \ldots, x_i; \sigma^i \omega) =$$

$$\frac{1}{n} \sum_{i=1}^{n+m-1} \log L((\tau^i x)_{-m+1}, \ldots, (\tau^i x)_0; \sigma^0(\sigma^i \omega)).$$

Since $Q$ is $(\tau \otimes \sigma)$-invariant and condition (2.1) holds, the result follows now from Birkhoff’s Ergodic Theorem. □

For every $\beta > 0$ and $n \in \mathbb{N}$ we put

$$T_n^{(\beta)}(\omega) := \sum_{x_{2-m}, \ldots, x_n} \prod_{i=1}^{n} L^\beta(x_{i-m+1}, \ldots, x_i; \sigma^i \omega).$$

Let $V = \{1, \ldots, n\}$. Note that $T_{n+m-1}^{(\beta)}(\omega) = \sum_{x_{2-m} \in X_{2-m}} \sigma_{n+m}^{(\beta)}(x; \omega)$.

Lemma 3.1 The following limit exists for $P$-almost all $\omega$

$$P(\beta) := \lim_{n \to \infty} \frac{1}{n} \log T_n^{(\beta)}(\omega), \quad \beta > 0. \quad (3.3)$$

Proof. One can easily check

$$T_k^{(\beta)}(\omega) T_n^{(\beta)}(\sigma^k \omega) =$$

$$\left( \sum_{y_{2-m} \ldots y_k} \prod_{i=1}^{k} L^\beta(y_{i-m+1}, \ldots, y_i; \sigma^i \omega) \right) \left( \sum_{z_{2-m} \ldots z_{k+n}} \prod_{i=k+1}^{k+n} L^\beta(z_{i-m+1}, \ldots, z_i; \sigma^i \omega) \right) \geq$$

$$\sum_{x_{2-m}, \ldots, x_{k+n}} \prod_{i=1}^{k+n} L^\beta(x_{i-m+1}, \ldots, x_i; \sigma^i \omega) = T_{k+n}^{(\beta)}(\omega).$$

Thus $\log T_{k+n}^{(\beta)}(\omega) \leq \log T_k^{(\beta)}(\omega) + \log T_n^{(\beta)}(\sigma^k \omega)$. Kingman’s Subadditive Ergodic Theorem (see, e.g., [13]) implies that the limit (3.3) exists and is $\sigma$-invariant and thus constant for almost all $\omega$. □

Theorem 3.2 For any $\beta > 0$ with $P$-probability one

$$\lim_{n \to \infty} \frac{1}{n} \log Z_n^{(\beta)}(x; \omega) = P(\beta) \quad \text{for all } x \in X, \quad (3.4)$$

where $P(\beta)$ is defined in (3.3).

Proof. Choose $\omega \in \Omega$ such that

$$\|U(\cdot \cdot \cdot \omega)\| < \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log T_n^{(\beta)}(\omega) = P(\beta).$$
Let
\[ G_n := \max_{x, y \in \partial V} \frac{Z_n^{(\beta)}(x; \omega)}{Z_n^{(\beta)}(y; \omega)} = \max_{x, y \in \partial V} \frac{Z_n^{(\beta)}(x; \omega)}{\min_{x, y \in \partial V} Z_n^{(\beta)}(x; \omega)}. \]
It is shown (see, e.g., [20]) that
\[ \lim_{n \to \infty} \frac{1}{n} \log G_n = 0. \]
(3.5)

The theorem follows now from Lemma 3.1 and the following inequality
\[ N^{2(m-1)} \min_{x \in \partial V} Z_n^{(\beta)}(x; \omega) \leq T^{(\beta)}_{n+m-1}(\omega) \leq N^{2(m-1)} \max_{x \in \partial V} Z_n^{(\beta)}(x; \omega). \]

4 Properties of \( P(\beta) \)

Lemma 4.1 \( P(\beta) \) is a continuous function of \( \beta > 0 \) such that
\[ \lim_{\beta \to 0} P(\beta) = \log N > 0, \quad \lim_{\beta \to \infty} P(\beta) = -\infty, \]
and thus equation \( P(\beta) = 0 \) has a positive solution \( \alpha < \infty \).

Proof. Let \( \beta_1, \beta_2 > 0 \). Fix \( \omega \in \Omega \) such that
\[ P(\beta) = \lim_{n \to \infty} \frac{1}{n} \log Z_n^{(\beta)}(x; \omega) \]
holds with \( \beta = \beta_1 \) and \( \beta = \beta_2 \). By Theorem 3.2 such \( \omega \) exists. We have (see, e.g., [19])
\[ |\log Z_n^{(\beta_1)}(x; \omega) - \log Z_n^{(\beta_2)}(x; \omega)| \leq n|\beta_1 - \beta_2|\|U(\cdot, \cdot, \omega)\|. \]
(4.1)
Now if we divide both sides of 4.1 by \( n \) and let \( n \) tend to \( \infty \) we find that for some constant \( 0 < C < \infty \)
\[ |P(\beta_1) - P(\beta_2)| \leq C|\beta_1 - \beta_2|. \]
This finishes the proof. \( \square \)

We notice that in the terminology of Statistical Mechanics \( P(\beta) \) is the pressure corresponding to the inverse temperature \( \beta \). In [12],[18],[21] it was shown that in the case \( m = 1 \)
\[ P(\beta) = E \left[ \log \sum_{p \in S} L^\beta(p; \omega) \right]. \]
If \( m \geq 2 \) the number \( P(\beta) \) can be identified as follows. Consider the space of real \( S^{m-1} \times S^{m-1} \) matrices \( A = (a(p; q))_{p, q \in S^{m-1}} \). Let \( A^* \) denote the conjugate matrix and \( ||A|| \) denote the \( l^1 \)-norm, i.e.,
\[ ||A|| := \sum_{p, q \in S^{m-1}} |a(p; q)|. \]
Let \( \beta > 0, \omega \in \Omega \) and \( x \in X \). Put
\[
v_{\omega}^{(\beta)}(x_{2-m}, \ldots, x_0; x_1, \ldots, x_{m-1}) := \prod_{i=1}^{m-1} L^{(\beta)}(x_{i-m+1}, \ldots, x_i; \sigma^i \omega).
\]

We denote random matrix \((v_{\omega}^{(\beta)}(p; q))_{p,q\in S^{m-1}}\) by \(V^{(\beta)}(\omega)\). For each \( k \in \mathbb{N} \) we put
\[
W_k^{(\beta)}(\omega) := \prod_{i=0}^{k-1} V^{(\beta)}(\sigma^{(m-1)} \omega).
\]

Then \( Z_k^{(\beta)}(x; \omega) \) is the \((x_{2-m}, \ldots, x_0; x_1, \ldots, x_{m-1})\)-th entry of matrix \(W_k^{(\beta)}(\omega)\). The following is an immediate consequence of Theorem 3.2 and equation (3.5).

**Theorem 4.1** For any \( \beta > 0 \) and \( \mathbb{P} \)-almost all \( \omega \)
\[
\lim_{k \to \infty} k^{-1} \log \| W_k^{(\beta)}(\omega) \| = P^{(\beta)}.
\]

By Oseledeč’s theorem (see, e.g., [13]) the limit
\[
\Lambda^{(\beta)}(\omega) := \lim_{k \to \infty} \left( (W_k^{(\beta)}(\omega))^* W_k^{(\beta)}(\omega) \right)^{1/2k}
\]
exists with \( \mathbb{P} \)-probability 1; \( \Lambda^{(\beta)}(\omega) \) is a random \( S^{m-1} \times S^{m-1} \) matrix such that all its eigenvalues are \( \sigma \)-invariant and, therefore, \( \mathbb{P} \)-a.s. constant. The following result is easily found using Oseledeč’s theorem.

**Theorem 4.2** Let \( \kappa^{(\beta)} \) be the spectral radius of matrix \( \Lambda^{(\beta)}(\omega) \). Then for any \( \beta > 0 \)
\[
P^{(\beta)} = \log \kappa^{(\beta)}.
\]

## 5 Hausdorff Dimension of Sequence Space with Respect to Random Scale Metric

Along with the space \( X = S^Z \) of two-sided sequences we consider the space \( X^+ = S^\mathbb{N} \) of one-sided sequences. Let \( K^+ \) be the family of the finite subsets of \( \mathbb{N} \), \( A^+ \) and \( T^+ \) denote the \( \sigma \)-algebra and the topology in \( X^+ \) generated by the cylinders respectively.

Let \( x \in X \). We denote by \( x^+ \) the ”projection” of \( x \) on \( X^+ \) such that \( x_i^+ = x_i, i \in \mathbb{N} \). Assume \( \beta > 0 \) and \( \omega \in \Omega \). The measure \( \mu^{(\beta)}(\omega) \) on \( A \) naturally defines the measure on \( A^+ \). For the later we will use the same notation and the definition is given by
\[
\mu^{(\beta)}(\omega)(C_V(x^+)) = \mu^{(\beta)}(\omega)(C_V(x))
\]
where \( x \in X \) and \( V \in K^+ \).
We shall remind here the definition of the Moran index. Fix $\omega \in \Omega$. Let $m \geq 1$ be an integer. An integer $k(r, x) \geq m$ is defined in the following way

$$
\begin{aligned}
& a_{k(r, x)+1}(x, \omega) \leq r \leq a_{k(r, x)}(x, \omega), \quad \text{if } a_m(x, \omega) \geq r, \\
& k(r, x) = m, \quad \text{if } a_m(x, \omega) < r.
\end{aligned}
$$

(5.1)

Let $\rho$ be a metric in space $X^+$. The Moran index of $\{a_n\}$ with respect to $\rho$ and the constant $b$ is the minimal number $\iota_\omega(b)$ with the following property: for any $x \in X^+$ and any $n \geq m$ there exist at most $\iota_\omega(b)$ pairwise disjoint cylinders $C_{m,k(a_n(x,\omega),y^{(i)})}(y^{(i)})$, $y^{(i)} \in X^+$, such that

$$
B(x, ba_n(x,\omega)) \cap C_{m,k(a_n(x,\omega),y^{(i)})}(y^{(i)}) \neq \emptyset
$$

If such a number $\iota_\omega(b)$ does not exist we put $\iota_\omega(b) = \infty$.

We call a family of functions $\{\rho_\omega(x^+,y^+), \omega \in \Omega\}$ on $X^+ \times X^+$ a random scale metric in $X^+$ with scale coefficients $L(x_1, \ldots, x_m; \omega)$ if for almost every $\omega \in \Omega$

$$
diam(C_n(x^+))_{\rho_\omega} \leq \tilde{a}_n(x^+; \omega) \quad \text{for all } x^+ \in X^+, \quad (5.2)
$$

and

$$
\iota_\omega(b) < \infty \quad \text{for some } b = b(\omega) > 0. \quad (5.3)
$$

In the sequel we assume that the following condition holds.

$$
\text{ess sup}_{\omega \in \Omega} L(x_1, \ldots, x_m; \omega) < 1 \quad x_1, \ldots, x_m \in S \quad (5.4)
$$

By the terminology used in [21], condition (5.4) and Theorem 3.1 mean that for almost every $\omega$ and every $\beta > 0$ the metric $\rho_\omega$ is $\mu_{\omega}^{(\beta)}$-averageable.

Fix $\omega \in \Omega$, $\beta > 0$, and let $x \in X$. The cylinder-wise local dimension of $\mu_\omega$ at point $x^+ \in X^+$ is defined by

$$
d_{\rho_\omega}(x^+; \omega) := \lim_{n \to \infty} \frac{\log \mu_{\omega}^{(\beta)}(C_n(x^+))}{\log \tilde{a}_n(x^+; \omega)} = \lim_{n \to \infty} \frac{\log \mu_{\omega}^{(\beta)}(C_n(x))}{\log \tilde{a}_n(x, \omega)}.
$$

Let $V = \{1, \ldots, n\}$. It is shown in [20, Chapter 8, Lemma 8.1] that in the case of a Gibbs measure we can use the conditional measure of the cylinder in the definition of the local dimension. More precisely, for almost all $\omega$

$$
d_{\rho_\omega}(x^+; \omega) = \lim_n \frac{\log \mu_{\omega}^{(\beta)}(C_V(x) \mid C_{\partial V}(x))}{\log a_n(x, \omega)} \quad \text{for all } x \in X.
$$

Let $A \subset X^+$ and $\rho$ be a metric in $X^+$. By $\dim_\rho A$ we denote the Hausdorff dimension of the set $A$ with respect to the metric $\rho$. We compute $\dim_{\rho_\omega} X^+$ using the local cylinder-wise dimension of the measure $\mu_{\omega}^{(\alpha)}$.

**Theorem 5.1** Assume $\rho_\omega$ is a random scale metric, $\alpha$ is defined in Lemma 4.1. If conditions (2.2), (5.2), (5.3) and (5.4) are fulfilled then

$$
\dim_{\rho_\omega} X^+ = \alpha \quad \text{for } P\text{-almost all } \omega \in \Omega.
$$
Proof. Conditions (2.2) and (5.4) imply that for all $\omega \in \Omega$ and $x \in X$

$$\limsup_{n \to \infty} \frac{1}{n} \log a_n(x, \omega) = \limsup_{n \to \infty} \frac{1}{n} \log \tilde{a}_n(x, \omega) < 0,$$

and

$$\lim_{n \to \infty} \frac{\log a_n(x, \omega)}{\tilde{a}_n(x, \omega)} = 1$$

(5.5)

Let $V = \{1, \ldots, n\}$. Since by Lemma 3.2 $\lim_n \frac{1}{n} \log Z_n(\omega) = P(\alpha) = 0$, we have

$$d_{\mu(\omega)}(x^+; \omega) = \lim_n \frac{\log \mu_n(C_V(x) \cap C_{\partial V}(x))}{\log a_n(x, \omega)}$$

(5.6)

$$= \alpha - \lim \frac{\frac{1}{n} \log Z_n(\omega)}{\frac{1}{n} \log a_n(x, \omega)} = \alpha,$$

which holds for $P$-almost every $\omega$ and, according to (3.4), for all $x \in X$.

Finally, Theorem 2.1 in [21], Lemmas 3.1 imply that for almost every $\omega \in \Omega$ and $\mu(\omega)$-almost all $x^+ \in X^+$

$$\dim_{\mu(\omega)} X^+ = d_{\mu(\omega)}(x^+; \omega) = \alpha.$$

□

6 Fractals in General Metric Spaces

The results of the previous section can be generalized to fractals in general complete metric spaces. For the details we refer the reader to [6] and [21].

Assume that $(\Omega, \mathcal{G}, \mathbb{P})$ is a probability space and $\sigma$ is an ergodic endomorphism of $(\Omega, \mathcal{G}, \mathbb{P})$. Let us consider a set $M$ endowed with a family of metrics $\{\lambda_\omega, \omega \in \Omega\}$; we assume that $(M, \lambda_\omega)$ is a complete metric space for any $\omega \in \Omega$. Recall that $S^*$ denote the set of all finite sequences of elements of $S$ and let $I(\omega) = \{I_{x^+}(\omega) : x^+ \in S^*\}$ be a random construction consisting of closed bounded subsets for each $\omega \in \Omega$; denote by $F(\omega)$ the generated fractal:

$$F(\omega) = \bigcap_{n=1}^\infty \bigcup_{x^+ \in S^n} I_{x^+}(\omega).$$

Assume that the construction $I(\omega)$ satisfies the following conditions. If $x^*$ and $y^*$ are distinct finite sequences of the same length, then

$$I_{x^*}(\omega) \cap I_{y^*}(\omega) \cap F(\omega) = \emptyset, \quad \text{for almost all } \omega \in \Omega$$

(6.1)

As in the previous sections we consider random contraction coefficients $L(x_1, \ldots, x_m; \omega)$, such that

$$0 < L(x_1, \ldots, x_m; \omega) < 1, \quad x_1, \ldots, x_m \in S, \quad \omega \in \Omega.$$

Let us assume that

$$\text{diam}_\omega(I_{[x|n]}(\omega)) \leq \tilde{a}_n(x, \omega), \quad \text{for almost all } \omega \text{ and all } x \in X^+.$$
Let $x \in X^+$. By $[x|k]$ we denote the restriction of the sequence $x$ to the first $k$ components. We generalize here the notion of Moran index introduced in Section 2. Let $x \in X^+$ and $r > 0$. The number $k(r, x)$ is defined in (5.1).

Fix $\omega \in \Omega$. The Moran index of the construction $I(\omega)$ corresponding to the constant $b > 0$ is the minimal number $\iota_\omega(b)$ with the following property: for any $s \in M$, any $x \in X^+$, and any $n \geq m$ there exist at most $\iota_\omega(b)$ pairwise disjoint sets $I_{[x|k]}(I_{[a_n(x, \omega), x|n]})(\omega)$, where $x^{(j)} \in X^+$, such that

$$B_\omega(s, ba_n(x, \omega)) \cap I_{[x|k]}(I_{[a_n(x, \omega), x|n]})(\omega) \neq \emptyset$$

If such a number $\iota_\omega(b)$ does not exist we put $\iota_\omega(b) = \infty$.

Consider the following condition.

$$\iota_\omega(b) < \infty \quad \text{for almost every } \omega \text{ and some } b = b(\omega) > 0. \quad (6.3)$$

For each $\omega$ we consider the ”coding” map $\phi_\omega$ of $F(\omega)$ onto $X^+$. If $\lambda^*_\omega$ is the restriction of the metric $\lambda_\omega$ to the set $F(\omega)$, then denote $\rho_\omega = \lambda^*_\omega \circ \phi_\omega^{-1}$. The family of functions $\{\rho_\omega(x, y) : \omega \in \Omega\}$ on $X^+ \times X^+$ is a random scale metric on $X^+$.

For almost every $\omega \in \Omega$ this map $\phi_\omega$ is an isometry of $(F(\omega), \lambda^*_\omega)$ onto $(X^+, \rho_\omega)$, which allows us to reduce the study of the dimension of the fractal $F(\omega)$ to the dimension of $X^+$ with respect to $\rho_\omega$. Using Theorem 5.1, we obtain the following result.

**Theorem 6.1** Assume that the random construction $I(\omega)$ satisfies the conditions (6.1), (6.2), and (6.3). If the contraction coefficients satisfy (2.2) and (5.4), and $\alpha$ is the number defined in Lemma 4.1, then with $P$-probability 1

$$\dim_{\lambda_\omega} F(\omega) = \alpha.$$

**7 Examples**

Consider two rather general examples where the pressure $P(\beta)$ can be computed more or less explicitly.

**Example 7.1** The deterministic case has been studied in [1] and [21]. In this case the matrices $V^{(\beta)} := V^{(\beta)}(\omega)$ and $W^{(\beta)} := W^{(\beta)}(\omega)$ are non-random and we have

$$W^{(\beta)}_k = (V^{(\beta)})^{k+1},$$

where $k = 1, 2, \ldots$. Let $R(\beta)$ be the spectral radius of $V^{(\beta)}$. From (4.2) it follows that $P(\beta) = \log R(\beta)$. Thus the Hausdorff dimension of the generated fractals is the solution of the equation $R(\beta) = 1$.

**Example 7.2** Consider a more general situation of a periodic dynamical system. Assume that $\sigma^{(m-1)}q_\omega = \omega$ for some $q \in \mathbb{N}$ and all $\omega \in \Omega$. Denote

$$V^{(\beta)}(\omega) := \prod_{i=0}^{q-1} V^{(\beta)}(\sigma^{(m-1)}\omega).$$
Then
\[ W^{(\beta)}_{nq-1} = (\tilde{V}^{(\beta)}(\omega))^n, \]
where \( n = 1, 2, \ldots \). Hence \( P(\beta) = \log R(\beta) \), where \( R(\beta) \) is the (non-random) spectral radius of matrix \( \tilde{V}^{(\beta)}(\omega) \).

References


