APPROXIMATION OF LATTICE RANDOM FIELDS BY MARKOV FIELDS AND HAUSDORFF DIMENSION OF THE SET OF GENERIC CONFIGURATIONS

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March 10, 2001
1. **Statement of results.** We consider strictly homogeneous random fields on the lattice $\mathbb{Z}^d$, $d \geq 1$, and identify every such field with a translation invariant probability measure on the configuration space $X = V^{\mathbb{Z}^d} = \{ x : \mathbb{Z}^d \to V \}$, where $V$ is the set of values of the field. We throughout assume that $V$ is finite and for short write $\mathbb{Z}^d =: T$. Endow the set $X$ with the product topology corresponding to the discrete topology on $V$ and define the group $\tau$ of transformations $\tau_t$, where $(\tau_t x)(s) = x(s + t)$, $x \in X$, $s, t \in T$. Introduce the set $I$ of $\tau$-invariant Borel probability measures on $X$. Let $T_n = [-n, n]^d$, $n = 0, 1, \ldots$ A point $x \in X$ is said to be generic with respect to a measure $\mu \in I$ ($\mu$-generic) if for every continuous function $f$ on $X$ ($f \in \mathcal{C}(X)$) we have

$$\lim_{n \to \infty} \frac{1}{|T_n|} \sum_{t \in T_n} f(\tau_t x) = \mu(f),$$

where $\mu(f) := \int f d\mu$ and $|A|$ is the cardinality of a finite set $A$. Denote the set of all $\mu$-generic points by $X_\mu$. 

Our goal in this paper is to evaluate, for any $\mu \in I$, the Hausdorff dimension $\dim_{H,\rho}(X_\mu)$ of $X_\mu$ with respect to any metric $\rho$ from a rather wide class of metrics on $X$ introduced in [14] and called scale metrics (see also [4], [5]). We recall here only the definition of the simplest metric in this class. We call his metric standard and denote it by $\rho_\theta$, where $\theta \in (0, 1)$. It is defined by

$$\rho_\theta(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \theta^{|n(x, y)|}, & \text{if } x \neq y, \end{cases}$$

where $n(x, y) = \min\{n : x(t) \neq y(t) \text{ for some } t \in T_n\}$

Our main result is the following

**Theorem 1.** For every scale metric and every $\mu \in I$,

$$\dim_{H,\rho}(X_\mu) = h(\mu)/\gamma(\rho),$$

where $h(\mu)$ is the entropy of $\mu$ and $\gamma(\rho)$ a constant (independent of $\mu$). In particular, $\gamma(\rho_\theta) = -\ln \theta$.

Thus Theorem 1 extends a similar result for Gibbs measures established in [7]. If $d = 1$ this result is contained in [2],[8].
Let us note that if \( \mu \) is an ergodic measure, it is supported by \( X_\mu \), but otherwise \( \mu(X_\mu) = 0 \). Nevertheless the expression for \( \dim_{H,\rho}(X_\mu) \) does not depend on the ergodicity.

The rest of the paper is devoted to the proof of Theorem 1. We begin with some notions and results of thermodynamic formalism.

2. Thermodynamic formalism (see [10], [3]). For any \( x \in X \) and \( \Lambda \subset T \), denote by \( x_\Lambda \) the restriction of \( x \) to \( \Lambda \). Let \( \mathcal{F} \) be the family of all finite subsets of \( T \). A function \( \Phi : \mathcal{F} \times X \to \mathbb{R} \) is said to be a potential if \( \Phi(F, x) = \Phi(F, y) \) as soon as \( x_F = y_F \); in the sequel we assume that the potential \( U \) is invariant, i.e. \( U(F + t, x) = U(F, \tau_t x), x \in X, s, t \in T, F \in \mathcal{F} \). The following two norms can be defined for potentials:

\[
||\Phi||_1 = \sum_{\Lambda \in \mathcal{F}, 0 \in \Lambda} \max_{x \in X} |\Lambda|^{-1} |\Phi(\Lambda, x)|,
\]

\[
||\Phi||_2 = \sum_{\Lambda \in \mathcal{F}, 0 \in \Lambda} \max_{x \in X} |\Phi(\Lambda, x)|.
\]

Let \( B_1 \) be the set of potentials \( \Phi \) with \( ||\Phi||_i < \infty, i = 1, 2 \). It is easy to see that \( B_1, B_2 \) are Banach spaces and \( B_2 \) is dense in \( B_1 \) (in the \( || \cdot ||_1 \) norm). A potential \( \Phi \) is said to be of finite range if there is a number \( r > 0 \) such that \( \Phi(\Lambda, x) \equiv 0 \) for all \( \Lambda \) with \( \text{diam}(\Lambda) \geq r \). To any potential \( \Phi \in B_1 \) one can assign the function \( f_\Phi \in C(X) \) (called the local energy) by

\[
f_\Phi(x) = \sum_{\Lambda \in \mathcal{F}, 0 \in \Lambda} |\Lambda|^{-1} \Phi(\Lambda, x), \ x \in X,
\]

and the map of \( B_1 \) to \( C(X) \) thus induced is surjective. The function \( P : B_1 \to \mathbb{R} \) defined by

\[
P(\Phi) = \sup_{\mu \in \mathcal{I}} [h(\mu) - \mu(f_\Phi)]
\]

is called the pressure of \( \Phi \). This function is finite and convex and hence continuous on \( B_1 \). It can be defined directly in terms of \( \Phi \) without resort to measures. In view of this fact relation (3) is referred to as the variational principle for \( P(\Phi) \). The variational principle for \( h(\mu) \) ”conjugate to” (3) also holds:

\[
h(\mu) = \inf_{\Phi \in B_1} [P(\Phi) + \mu(f_\Phi)] = \inf_{\Phi \in B_2} [P(\Phi) + \mu(f_\Phi)], \ \mu \in \mathcal{I}.
\]

A measure \( \mu_0 \in \mathcal{I} \) is said to be \( \Phi \)-equilibrium if the supremum in (3) is attained at \( \mu_0 \). We denote the set of \( \Phi \)-equilibrium measures by \( \mathcal{E}(\Phi) \).
Remark 1. Every measure $\nu \in \mathcal{E}(\Phi)$ with $\Phi \in \mathcal{B}_2$ is a Gibbs measure. If $\Phi$ is of finite range such a $\nu$ is a Markov measure and the corresponding random field is Markovian.

3. Upper bound of $\dim_{H,\rho}(X, \mu)$. For any $\Lambda \in \mathcal{F}$ and $x \in X$ we introduce the cylinder set

$$C_\Lambda(x) := \{y \in X : y_\Lambda = x_\Lambda\}, \quad x \in X.$$  

We shall write $C_n(x)$ instead of $C_{T_n}(x)$. For a Borel probability measure $\nu$ on $X$, define the upper and lower local cylinder entropies of $\nu$ by

$$d_\nu(x) = \lim_{n \to \infty} \frac{1}{|T_n|} \ln \nu(C_n(x)), \quad \bar{d}_\nu(x) = \lim_{n \to \infty} \frac{1}{|T_n|} \ln \nu(C_n(x))$$

(let us note that $\nu(C_n(x)) > 0$ for $\nu$-almost all $x \in X$). The main tool we use for estimating the Hausdorff dimension of sets is the following version of Frostman’s lemma (cf. [14], [5]).

Lemma 1. Let $\nu$ be a Borel probability measure on $X$, $\rho$ a scale metric on $X$, and $A \subset X$. Then

(a) $\dim_{H,\rho}(A) = \sup_{x \in A} \bar{d}_\nu(x)/\gamma(\rho)$,
(b) $\dim_{H,\rho}(A) \geq \inf_{x \in A} d_\nu(x)/\gamma(\rho)$, provided that $\nu(A) > 0$.

For a function $f \in C(X)$ and a measure $\mu \in \mathcal{I}$, we set

$$X_{\mu,f} = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{|T_n|} \sum_{t \in T_n} f(t_\tau x) = \mu(f) \right\}.$$  

Lemma 2. (see [7]). Let $\mu \in \mathcal{I}$, $\Phi \in \mathcal{B}_2$, and $\nu \in \mathcal{E}(\Phi)$. Then

$$\bar{d}_\nu(x) = d_\nu(x) = P(\Phi) + \mu(f_\Phi)$$

for every $x \in X_{\mu,f_\Phi}$.

Proposition 1. For every $\mu \in \mathcal{I}$ and every scale metric $\rho$

$$\dim_{H,\rho}(X_\mu) \leq h(\mu)/\gamma(\rho). \quad (5)$$
Proof. By Lemma 1(a) and Lemma 2
\[
\dim_{H,\rho}(X_{f,\Phi}) \leq \frac{1}{\gamma(\rho)} [P(\Phi) + \mu(\Phi)]
\]
for every \( \Phi \in B_2 \). Therefore
\[
\dim_{H,\rho}(X_{\mu}) \leq \frac{1}{\gamma(\rho)} \inf_{\Phi \in B_2} [P(\Phi) + \mu(\Phi)],
\]
and (5) follows immediately from (4).

4. Lower bound of \( \dim_{H,\rho}(X, \mu) \): ergodic case. If the measure \( \mu \) is \( \tau \)-ergodic, it is quite easy to obtain a good estimate from below for \( \dim_{H,\rho}(X, \mu) \).

Indeed, in this case \( \mu(X_{\mu}) = 1 \), and by the \( d \)-dimensional version of the Shannon–Macmillan–Breiman theorem [9] \( d_\mu(x) = \overline{d}_\mu(x) = h(\mu) \) for \( \mu \)-almost all \( x \in X_{\mu} \). Hence from Lemma 1(b) (where \( \mu \) is taken for \( \nu \)) it follows that
\[
\dim_{H,\rho}(X_{\mu}) \geq h(\mu)/\gamma(\rho).
\]
(6)

Now, taking into account Proposition 1, we come to (1).

5. Well-approximated measures. We now consider the opposite case where \( \mu \) is not ergodic. Here we cannot use \( \mu \) as \( \nu \) in Lemma 1(b), since the condition \( \mu(X_{\mu}) > 0 \) fails. On the other hand, in [7] we proved that the lower estimate (6) does hold for a class of measures \( \mu \) which do not need to be ergodic. Let us describe this class of ”well-approximated” measures (below we prove that, in fact, it coincides with the whole class \( \mathcal{I} \)).

We say that a measure \( \nu \in \mathcal{I} \) is quasi-Bernoullian if there exists a sequence of numbers \( u_n \geq 0 \), \( n = 1, 2, \ldots \), such that for each positive integer \( n \), each pair \( x, y \in X \), and each \( \Lambda \in \mathcal{F} \), \( \Lambda \subset T \setminus T_n \), we have
\[
\nu(C_n(x) \cap C_\Lambda(y)) \leq \nu(C_n(x))\nu(C_\Lambda(y)) \exp(u_n|T_n|).
\]

Proposition 2. (see [7]). Every measure \( \nu \in \mathcal{E}(\Phi) \) with \( \Phi \in B_2 \) is quasi-Bernoullian.

We say that a measure \( \nu \in \mathcal{I} \) is well-approximated if there is a sequence of \( \tau \)-ergodic quasi-Bernoullian measures \( \mu_n \in \mathcal{I} \) such that
\[
\mu_n \to \mu \text{ and } h(\mu_n) \to h(\mu) \text{ as } n \to \infty
\]
(7)
(here as well as in the sequel we consider only weak convergence of measures).

The following statement is actually proved in [7].
Proposition 3. Inequality (6) holds for every well-approximated measure \( \mu \in \mathcal{I} \).

We now see that for establishing Theorem 1 it remains to prove the following

Theorem 2. Each measure in \( \mathcal{I} \) is well-approximated.

6. Markov approximation of measures. Instead of Theorem 2 we prove in this section the following, a little stronger statement, which may of interest in its own right.

Theorem 3. For every measure \( \mu \in \mathcal{I} \), there is a sequence of finite range potentials \( \Phi_n \) and a sequence of \( \tau \)-ergodic (Markov) measures \( \mu_n \in \mathcal{E}(\Phi_n) \) such that (7) holds.

In the proof of Theorem 3 we use a well-known fact of convex analysis. For the reader’s convenience we recall underlying notions and state this fact as a lemma.

Let \( \varphi \) be a convex function on a Banach space \( B \). A linear functional \( l \in B^* \) is called \( \varphi \)-bounded if there is \( c \in \mathbb{R} \) such that \( l(U) - c \leq \varphi(U) \) for all \( U \in B \). A functional \( l \in B^* \) is called tangent to \( \varphi \) at the point \( U_0 \in B \) (we write \( l \in \partial \varphi(U_0) \)) if

\[
\varphi(U_0 + U) \geq \varphi(U_0) + l(U)
\]

for all \( U \in B \). Every tangent functional to \( \varphi \) is obviously \( \varphi \)-bounded.

Lemma 3. (Bishop–Phelps Theorem [1]) Let \( B, \varphi \) be as above and \( U_0 \in B \), \( l_0 \in B^* \), \( l_0 \) be \( \varphi \)-bounded. Then for each \( \varepsilon > 0 \) there are \( U \in B \) and \( l \in B^* \) such that (a) \( l \in \partial \varphi(U) \), (b) \( ||l - l_0||_{B^*} \leq \varepsilon \), (c) \( ||U - U_0||_B \leq \frac{1}{\varepsilon} [\varphi(U_0) - l_0(U_0) - s(l_0)] \), where

\[
s(l_0) = \inf_{V \in B} [\varphi(V) - l_0(V)].
\]

Proposition 4. Let \( \Phi \in B_1 \) and \( \mu \in \mathcal{E}(\Phi) \). Then there exist a sequence of potentials \( \Phi_n \in B_2 \) and a sequence of measures \( \mu_n \in \mathcal{E}(\Phi_n) \) such that

\[
(a) \lim_{n \to \infty} ||\Phi_n - \Phi||_1 = 0, \quad (b) \lim_{n \to \infty} \mu_n = \mu, \quad (c) \lim_{n \to \infty} h(\mu_n) = h(\mu). \quad (8)
\]
Proof. Given \( \delta > 0 \), one can find a potential \( \Phi_\delta \in B_2 \) such that \( ||\Phi - \Phi_\delta||_1 \leq \delta \) (\( \Phi_\delta \) can even be taken of finite range). Then (as is well known and can be easily checked; see [10], Sections 3.2, 3.4)

\[
||f_\Phi - f_{\Phi_\delta}|| = ||f_\Phi - f_{\Phi_\delta}||_1 \leq ||\Phi - \Phi_\delta||_1 \leq \delta
\]

(9)

(here \( || \cdot || \) is the supremum norm) and

\[
|P(\Phi) - P(\Phi_\delta)| \leq ||f_\Phi - f_{\Phi_\delta}|| \leq \delta.
\]

(10)

We define a functional \( l_0 \) on \( B_2 \) by

\[
l_0(\Psi) = -\mu(f_\Psi), \quad \Psi \in B_2.
\]

One can readily check that \( l_0 \in B_2^* \) and \( l_0 \) is \( P|_{B_2} \)-bounded (in the latter case we use the variational principle (3)). Then the assumptions of Lemma 3 are satisfied for \( B = B_2 \), \( \varphi = P|_{B_2} \), \( U_0 = \Phi_\delta \), and \( l_0 \) defined as above. Applying this lemma with \( \varepsilon = \delta^{1/2} \), we obtain a potential \( \Psi_\delta \in B_2 \) and a functional \( l_\delta \in B_2^* \) such that

\[
l_\delta \in \partial P(\Psi_\delta), \quad ||l_\delta - l_0||_{B_2^*} \leq \delta^{1/2},
\]

(11)

\[
||\Phi_\delta - \Psi_\delta||_2 \leq \delta^{-1/2}|P(\Phi_\delta) + \mu(f_{\Phi_\delta}) - s(l_0)|.
\]

(12)

By definition

\[
s(l_0) = \inf_{\Psi \in B_2} [P(\Psi) - l_0(\Psi)] = \inf_{\Psi \in B_2} [P(\Psi) + \mu(f_\Psi)].
\]

Hence (see (4))

\[
s(l_0) = h(\mu).
\]

(13)

Since \( \mu \in \mathcal{E}(\Phi) \), we have \( h(\mu) = P(\Phi) + \mu(f_\Phi) \). Together with (12), (13), (10) this implies

\[
||\Phi_\delta - \Psi_\delta||_{B_2} \leq \delta^{-1/2}[P(\Phi_\delta) - P(\Phi) + \mu(f_{\Phi_\delta} - f_\Phi)] \leq 2\delta^{1/2}.
\]

(14)

Let us now deal with the functional \( l_\delta \). It is known (see [3], Theorem (16.14)) that for each \( \Psi \in B_2 \), the map \( j : \nu \mapsto -\nu(f_\Psi) \) induces a 1-1-correspondence between \( \mathcal{E}(\Psi) \) and \( \partial P(\Psi) \). Hence there is a measure \( \mu_\delta \in \mathcal{E}(\Psi_\delta) \) such that

\[
l_\delta(\Psi) = \mu_\delta(f_\Psi) \text{ for all } \Psi \in B_2.
\]

(15)
Take an arbitrary sequence of numbers $\delta_n > 0$ with $\lim_{n \to \infty} \delta_n = 0$ and let

$$
\Psi_n := \Psi_{\delta_n}, \quad \mu_n := \mu_{\delta_n}, \quad l_n := l_{\delta_n}.
$$

(16)

From (9), (11), (14), (15), and (16) we obtain

$$
(a) \lim_{n \to \infty} ||\Phi_n - \Phi||_{B_1} = 0, \quad (b) \lim_{n \to \infty} ||l_n - l_0||_{B_2} = 0,
$$

(17)

where

$$
\mu_n(\Psi) = \mu_{\delta_n}(f_{\Psi}) \text{ for all } \Psi \in B_2, \ \mu_n \in \mathcal{E}(\Phi_n).
$$

Equality (17)(b) implies that

$$
\lim_{n \to \infty} l_n(\Psi) = l_0(\Psi)
$$

regardless of the explicit form of the norm $||\cdot||_*$, which means (see (16), (18)) that

$$
\lim_{n \to \infty} \mu_n(f_{\Psi}) = \mu(f_{\Psi}), \quad \Psi \in B_2.
$$

(19)

Since the functions of the form $f_{\Psi}$ where $\Psi$ is of finite range constitute a dense subset of $\mathcal{C}(X)$ (every function depending on $x_F$ with $F \in \mathcal{F}$ is of this form), (19) implies the weak convergence of $\mu_n$ to $\mu$.

It remains to proof that $h(\mu_n) \to h(\mu)$ as $n \to \infty$. By definition (recall that $\mu \in \mathcal{E}(\Phi)$, $\mu_n \in \mathcal{E}(\Phi_n)$)

$$
P(f_{\Phi_n}) = h(\mu_n) - \mu_n(f_{\Phi_n}), \quad P(f_{\Phi}) = h(\mu) - \mu(f_{\Phi}),
$$

so that

$$
h(\mu_n) - h(\mu) = P(f_{\Phi_n}) - P(f_{\Phi}) + \mu_n(f_{\Phi_n}) - \mu(f_{\Phi}),
$$

and the desired assertion follows immediately from (17)(a), the inequality

$$
|P(\Phi_n) - P(\Phi)| \leq ||\Phi_n - \Phi||_1,
$$

and the convergence $\mu_n \to \mu$. \square

We now can add one more item to the properties (a)-(c) in Proposition 4.

**Proposition 5.** Let $\Phi \in B_1$ and $\mu \in \mathcal{E}(\Phi)$. Then there are $\Phi'_n \in B_2$ and $\mu'_n \in \mathcal{E}(\Phi'_n)$, $n = 1, 2, \ldots$, such that $E(\Phi'_n) = \{\mu'_n\}$ and equalities $8(a)-(c)$ hold (with $\Phi_n$, $\mu_n$ replaced by $\Phi'_n$, $\mu'_n$).
Proof. Let $\Phi_n$ and $\mu_n$ be as in Proposition 4. By Sokal’s theorem ([3], Corollary (16.38)) for each $n$, there are potentials $\Phi_{n,k} \in B_2$ and measures $\mu_{n,k} \in E(\Phi_{n,k})$, $k = 1, 2, \ldots$, such that

(a) $\lim_{k \to \infty} ||\Phi_{n,k} - \Phi_n|| = 0$, (b) $\lim_{k \to \infty} \mu_{n,k} = \mu_n'$, (c) $E(\Phi'_n) = \{\mu'_{n,k}\}$.

Hence by putting $\Phi_n' := \Phi_{n,k_n}$, $\mu_n' := \mu'_{n,k_n}$, where $k_n$ increases sufficiently fast, we come to the desired result.

Proof of Theorem 3. According to Proposition 5 for every measure $\mu \in I$ one can find a close (in the weak topology) measure $\mu'$ with close entropy and such that $\mu'$ is a unique $\Phi'$-equilibrium measure for some $\Phi' \in B_2$. We can then take a finite range potential $\Phi''$ close to $\Phi'$ (in the $B_2$-norm) and an arbitrary ergodic measure $\mu'' \in E(\Phi'')$. This measure is close to $\mu'$ (because the latter is unique) and its entropy is close to $h(\mu')$ (for the same reason).

References


