SOLUTIONS TO HWK 6

PROBLEM A. 1). \( f(x; \theta) = \exp\left[-\frac{x}{\theta} + \ln x - 2 \ln \theta \right]. \)
2) \( K(x) = x, p(\theta) = -\frac{1}{\theta}; Q(x) = \ln x; q(\theta) - 2 \ln \theta. \)
The support \( S = (0, \infty) \) for all \( \theta \in \Omega = (0, \infty). \)
\( K'(x) = 1 \neq 0, p(\theta) \neq \text{const}\) on \( \Omega. \)
Hence the family \( f(x; \theta) \) is a regular exponential family.
3) \( Y = \sum_{i=1}^{n} X_i. \)
4) The MLE \( \hat{\theta} = \frac{\bar{X}}{2}; \mu = 2\theta; E(\hat{\theta}) = \theta. \) Hence \( \hat{\Theta} \) is unbiased.
6) By the theorem about unbiased functions of complete sufficient statistics \( \hat{\Theta} \) is the MVUE. We could also use directly the corollary about unbiased functions of MLEs (here this is the identity function because \( \hat{\Theta} \) is an unbiased estimator and we do not have to look for any other function).

6.2-2 (a) (77.272,92,782); (b) (79.12,90.88); (c) (80.065,89.935); (d) (81.154,88.846).
Note: when \( Q \) is decreasing, the CI become more narrow (we can afford more risk of a mistake, and we provide more information).

6.2-4 (a) \( \bar{x} = 56.8; \)
(b) \([56.8 - 1.96(\frac{2}{\sqrt{10}}), 56.8 + 1.96(\frac{2}{\sqrt{10}})] = [55.56, 58.04]. \)

Problem B 57 \( \in CI \) but 58.2 \( \notin CI; \) hence the first statement agrees with the data, but the second does not.

6.2-5 \( \bar{x} = 60.37, \ s = 39.62. \)
\[ \alpha = 0.1, \ z_{\alpha} = 1.645; \Delta_{0.05} = 1.645 \left(\frac{39.61}{\sqrt{30}}\right) = 11.9. \]
The approximate 90% CI is: \( \mu_E \in [48.47, 72.27]. \)
In fact, using our tables we can find the explicit 90% CI; indeed, \( t_{0.05}(29) = 1.697, \)
the explicit \( \Delta_{0.05} = 1.697 \left(\frac{39.61}{\sqrt{30}}\right) = 12.27, \) and the explicit 90% CI is \([48.1, 72.64]. \)

6.2-6 The approximate CI for \( \mu_W \) is:
\[ \left[11.95 - 1.96 \left(\frac{11.80}{\sqrt{37}}\right), 11.95 + 1.96 \left(\frac{11.80}{\sqrt{37}}\right)\right] = [8.15, 15.75]. \]
If more extensive \( t \)-tables are available or if a computer program is used we have the explicit CI
\[ \left[11.95 - 2.028 \left(\frac{11.80}{\sqrt{37}}\right), 11.95 + 2.028 \left(\frac{11.80}{\sqrt{37}}\right)\right] = [8.016, 15.884]. \]
6.2-10 From table VI \( t_{0.1}(27) = 1.314 \) Therefore the CI for the lower bound is 
\[
[21.45 - 1.314 \left( \frac{0.31}{\sqrt{28}} \right), \infty) = [21.373, \infty).
\]

**Problem C**

The CI \([8.15, 15.75]\) for \(\mu_W\) is located to the left of the CI \([48.1, 72.64]\) for \(\mu_E\), therefore \(\mu_W < \mu_E\).

**Problem D.** \(\mu = \left(\frac{1}{\theta}\right) \int_0^1 x \cdot x^{(1-\theta)/\theta} \, dx = \left(\frac{1}{\theta}\right) \int_0^1 x^{1/\theta} \, dx = \frac{1}{\theta+1}\). Solving the equation \(\mu = \frac{1}{\theta+1}\) for \(\theta\) we obtain \(\theta = \frac{1-\mu}{\mu}\). We substitute the first moment \(\mu_1 = \mu\) by its empirical estimator \(M_1 = \bar{X}\) and obtain the method-of-moments estimator for \(\theta\):
\[
\hat{\theta} = \frac{1 - \bar{X}}{\bar{X}}.
\]